Chapter 2: Continuity and limit of the function

Neighbourhood of a point

Notation: $\forall a \in \mathbb{R}\ \forall \varepsilon > 0$ we denote

- $\mathcal{O}_\varepsilon(a) = (a-\varepsilon, a+\varepsilon)$ ε-neighbourhood of a point a
  $\mathcal{O}^+_\varepsilon(a) = [a, a+\varepsilon)$ right ε-neighbourhood
  $\mathcal{O}^-_\varepsilon(a) = (a-\varepsilon, a]$ left ε-neighbourhood
- $\mathcal{P}_\varepsilon(a) = \mathcal{O}_\varepsilon(a) \setminus \{a\}$ punctured ε-neighbourhood of a point a
  $\mathcal{P}^+_\varepsilon(a) = (a, a+\varepsilon)$ punctured right ε-neighbourhood
  $\mathcal{P}^-_\varepsilon(a) = (a-\varepsilon, a)$punctured left ε-neighbourhood
- $x \to a$ $\quad$ x "tends to"a
  x takes values arbitrarily close to a

  $x \to a+,\ x \to a-,\ x \to +\infty,\ x \to -\infty$

Definition: Consider a function f defined in a neighbourhood of a. We say that f is continuous at point $a \in D(f)$ if

$$\forall \mathcal{O}_\varepsilon(f(a)) \, \exists \mathcal{O}_\delta(a) : \quad f(\mathcal{O}_\delta(a)) \subseteq \mathcal{O}_\varepsilon(f(a)).$$

Equivalently:

$$\forall \varepsilon > 0 \, \exists \delta > 0 : \quad |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Definition: We say that f is continuous on an open interval (a, b) if it is continuous at each point of (a, b).

Continuity of a function (2)

Definition: We say that a function f is continuous from the right (right-hand side continuous) /continuous from the left (left-hand side continuous) at point $a \in D(f)$ if

$$\forall \mathcal{O}_\varepsilon(f(a))\ \exists \mathcal{O}^+_\delta(a): \quad f(\mathcal{O}^+_\delta(a)) \subseteq \mathcal{O}_\varepsilon(f(a))$$

$$\forall \mathcal{O}_\varepsilon(f(a))\ \exists \mathcal{O}^-_\delta(a): \quad f(\mathcal{O}^-_\delta(a)) \subseteq \mathcal{O}_\varepsilon(f(a))$$

Definition: We say that a function f is continuous on a closed interval $[a, b]$ if it is

- continuous at each point of (a, b),
- continuous from the right at point a,
- continuous from the left at point b.

Continuity of a function (3)

Theorem: Let f and g be functions continuous at point a. Then functions $|f|$, $f \pm g$, $f \cdot g$ are continuous at point a. Furthermore if $g(a) \neq 0$ then $\frac{f}{g}$ is continuous at point a.

Theorem: If

- function $y = f(x)$ is continuous at point $x = a$,
- function $z = g(y)$ is continuous at point $y = f(a)$,

then a composition $h(x) = g(f(x))$ is continuous at point $x = a$.

Definition: Let $f : D(f) \to \mathbb{R}$ be defined in some neighbourhood $\mathcal{P}(a) \subseteq D(f)$. We say that function f has limit $A \in \mathbb{R}$ at point a (denote $\lim_{x \to a} f(x) = A$) if

$$\forall \mathcal{O}_\varepsilon(A)\ \exists \mathcal{P}_\delta(a) : \quad f(\mathcal{P}_\delta(a)) \subset \mathcal{O}_\varepsilon(A)$$

or equivalently

$$\forall \varepsilon > 0\ \exists \delta > 0 : \quad 0 < |x - a| < \delta \Rightarrow |f(x) - A| < \varepsilon$$

Theorem: Function f has at most one limit at point a.

Remark: We are interested in small ε (i.e. close to zero).

Calculation of the limits (1)

Theorem: Function f is continuous at point a if and only if $\lim_{x \to a} f(x) = f(a)$.

Theorem: Let $f : D(f) \to \mathbb{R}$, $g : D(g) \to \mathbb{R}$, $a \in \mathbb{R}$.

$$\exists \mathcal{P}(a) : \quad (\forall x \in \mathcal{P}(a) : \; f(x) = g(x)) \Rightarrow \lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

Squeeze theorem (Sandwich theorem):
Let the following two conditions hold:

- $\forall x \in \mathcal{P}(a) : \; g(x) \leq f(x) \leq h(x)$
- $\lim_{x \to a} g(x) = \lim_{x \to a} h(x)$

then there exists $\lim_{x \to a} f(x)$ and is equal to $\lim_{x \to a} g(x)$.

Calculation of the limits (2)

Theorem: Let $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, where $A, B \in \mathbb{R}$. It holds:

(i) $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$

(ii) $\lim_{x\to a}(f(x) cdot g(x)) = \lim_{x\to a} f(x). \lim_{x\to a} g(x) = A.B$

(iii) if $B \neq 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = (\lim_{x\to a} f(x))/(\lim_{x\to a} g(x)) = \frac{A}{B}$

Theorem (composition of functions): Let $\lim_{x\to a} g(x) = A$ and let the function f be continuous at point A. Then

$$\lim_{x\to a} f(g(x)) = f(A).$$

Remark:

$$\lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a} e^{g(x) \ln f(x)}$$

One sided limits

Definition: Let function $f : D(f) \to \mathbb{R}$ be defined on some $\mathcal{P}^+(a) \subseteq D(f)$. We say that f has right-hand limit $A \in \mathbb{R}$ at point a (symbolically $\lim\limits_{x \to a+} f(x) = A$) if

$$\forall \mathcal{O}_\varepsilon(A) \, \exists \mathcal{P}_\delta^+(a) : \quad f(\mathcal{P}_\delta^+(a)) \subset \mathcal{O}_\varepsilon(A)$$

Definition: Let function $f : D(f) \to \mathbb{R}$ be defined on some $\mathcal{P}^-(a) \subseteq D(f)$. We say that f has left-hand limit $A \in \mathbb{R}$ at point a (symbolically $\lim\limits_{x \to a-} f(x) = A$) if

$$\forall \mathcal{O}_\varepsilon(A) \, \exists \mathcal{P}_\delta^-(a) : \quad f(\mathcal{P}_\delta^-(a)) \subset \mathcal{O}_\varepsilon(A)$$

Theorem: $\lim\limits_{x \to a} f(x)$ exists if and only if $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$.
Then

$$\lim_{x \to a} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x).$$

Theorems for limits hold also for one-sided limits:

(i) f has at point a at most one left- and right-hand limit

(ii) f is left-hand (right-hand) side continuous at point a if and only if $\lim\limits_{x\to a\pm} f(x) = f(a)$

(iii) $f(x) = g(x)$ for $x \in \mathcal{P}^{\pm}(a) \Rightarrow \lim\limits_{x\to a\pm} f(x) = \lim\limits_{x\to a\pm} g(x)$

(iv) The squeeze theorem:

$$\left.\begin{array}{l}\forall x \in \mathcal{P}^{\pm}(a) : g(x) \leq f(x) \leq h(x) \\ \lim\limits_{x\to a\pm} g(x) = \lim\limits_{x\to a\pm} h(x)\end{array}\right. \Rightarrow \lim_{x\to a\pm} f(x) = \lim_{x\to a\pm} g(x)$$

(v) $\lim\limits_{x\to a\pm} (f(x)\dot{\pm}g(x)) = \lim\limits_{x\to a\pm} f(x)\dot{\pm} \lim\limits_{x\to a\pm} g(x)$

(vi) $\lim\limits_{x\to a\pm} \dfrac{f(x)}{g(x)} = (\lim\limits_{x\to a\pm} f(x))/(\lim\limits_{x\to a\pm} g(x))$ if $\lim\limits_{x\to a\pm} g(x) \neq 0$

(vii) $\lim\limits_{x\to a\pm} g(x) = A$
if f is right-(left-)hand side continuous at point A $\Rightarrow \lim\limits_{x\to a\pm} f(g(x)) = f(A)$

Let

$$\lim_{x\to a} f(x) = L.$$

Points $\pm\infty$ are called **improper points.** A point $a \in \mathbb{R}$ is called **proper point.**

I. If $a,\ L \in \mathbb{R}$ then L is a proper limit at proper point
II. If $a \in \mathbb{R},\ L = \pm\infty$ then L is a improper limit at proper point
III. If $a = \pm\infty\ L \in \mathbb{R}$ then L is a proper limit at improper point
IV. If $a = \pm\infty,\ L = \pm\infty$ then L is a improper limit at improper point

Case I. was discussed at previous section. Now we focused on cases II., III. a IV.

Improper Limits II.

Definition: Let $f(x)$ be defined on some $\mathcal{P}(a)$ then

(i) $\lim_{x\to a} f(x) = \infty$ if

$$\forall\ K > 0\ \ \exists\ \mathcal{P}_\delta(a)\ \text{ such that }\ \forall\ x \in \mathcal{P}_\delta(a)\ \text{ is }\ f(x) > K,$$

(ii) $\lim_{x\to a} f(x) = -\infty$ if

$$\forall\ L < 0\ \ \exists\ \mathcal{P}_\delta(a)\ \text{ such that }\ \forall\ x \in \mathcal{P}_\delta(a)\ \text{ is }\ f(x) < L.$$

Remark: Using $\mathcal{P}_\delta^+(a)$ or $\mathcal{P}_\delta^-(a)$ instead of $\mathcal{P}_\delta(a)$ we obtain one-hand improper limits at proper point:

(i) $\lim_{x\to a+} f(x) = \infty \qquad \lim_{x\to a-} f(x) = \infty$

(ii) $\lim_{x\to a+} f(x) = -\infty \qquad \lim_{x\to a-} f(x) = -\infty$

Theorem:

(i) $\lim_{x\to a} f(x) = \infty \quad \Leftrightarrow \quad \lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = \infty$

(ii) $\lim_{x\to a} f(x) = -\infty \quad \Leftrightarrow \quad \lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = -\infty$

Theorem:

(i) Let $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then

$$\lim_{x\to a} f(x) + g(x) = \infty \qquad \lim_{x\to a} f(x)\cdot g(x) = \infty\,.$$

(ii) Let $\lim_{x\to a} f(x) = -\infty$ and $\lim_{x\to a} g(x) = -\infty$ then

$$\lim_{x\to a} f(x) + g(x) = -\infty \qquad \lim_{x\to a} f(x)\cdot g(x) = \infty\,.$$

(iii) Let $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = -\infty$ then

$$\lim_{x\to a} f(x)\cdot g(x) = -\infty$$

(iv) Let $\lim_{x\to a} f(x) = A \in \mathbb{R}$, $A > 0$ and $\lim_{x\to a} g(x) = \pm\infty$ then

$$\lim_{x\to a} f(x)\cdot g(x) = \pm\infty\,.$$

(v) Let $\lim_{x\to a} f(x) = A \in \mathbb{R}$ and $\lim_{x\to a} g(x) = \pm\infty$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0\,.$$

Theorem: Let function f be bounded on some $P(a)$ then it holds

(i) If $\lim\limits_{x\to a} g(x) = \pm\infty$ then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = 0$.

(ii) If $\lim\limits_{x\to a} g(x) = 0$ then $\lim\limits_{x\to a} f(x)g(x) = 0$.

Theorem: Let $\lim\limits_{x\to a} f(x) = A > 0$ and $\lim\limits_{x\to a} g(x) = 0$ then

(i) If $g(x) > 0$ on $\mathcal{P}(a)$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = +\infty .$$

(ii) If $g(x) < 0$ on $\mathcal{P}(a)$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = -\infty .$$

(iii) If function $g(x)$ takes on each neighbourhood $P(a)$ positive and negative values then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \text{ does not exist.}$$

Definition: Consider f such that $(x_0, \infty) \subseteq \mathcal{D}(f)$ ($(-\infty, x_0) \subseteq \mathcal{D}(f)$, respectively). x_0 can be infinite.
We say that f has limit L_1 (L_2, respectively) at improper point ∞ ($-\infty$) and write

$$\lim_{x\to\infty} f(x) = L_1 \quad \left(\lim_{x\to-\infty} f(x) = L_2 \right)$$

if

$$\forall\, \mathcal{O}_\varepsilon(L_1)\ \exists\, x_1 > 0 \text{ such that } \forall\, x > x_1 \text{ is } f(x) \in \mathcal{O}_\varepsilon(L_1)$$

$$(\forall\, \mathcal{O}_\varepsilon(L_2)\ \exists\, x_2 < 0 \text{ such that } \forall\, x < x_2 \text{ is } f(x) \in \mathcal{O}_\varepsilon(L_2))$$

Squeeze theorem (Sandwich theorem):
Suppose:

- $\forall x \in (a, \infty): \; g(x) \leq f(x) \leq h(x)$
- $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x)$

then there exists $\lim_{x\to\infty} f(x)$ and is equal to $\lim_{x\to\infty} g(x)$.

Theorem: Let $\lim_{x\to\pm\infty} f(x) = A$ and $\lim_{x\to\pm\infty} g(x) = B$, where $A, B \in \mathbb{R}$. Then:

(i) $\lim_{x\to\pm\infty}(f(x) \pm g(x)) = \lim_{x\to\pm\infty} f(x) \pm \lim_{x\to\pm\infty} g(x) = A \pm B$

(ii) $\lim_{x\to\pm\infty}(f(x) \cdot g(x)) = \lim_{x\to\pm\infty} f(x) \cdot \lim_{x\to\pm\infty} g(x) = A \cdot B$

(iii) if $B \neq 0$ then $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\pm\infty} f(x)}{\lim_{x\to\pm\infty} g(x)} = \frac{A}{B}$

Improper Limits IV.

Definition: Consider f such that $(x_0, \infty) \subseteq \mathcal{D}(f)$ ($(-\infty, x_0) \subseteq \mathcal{D}(f)$, respectively). x_0 can be infinite.

(i) We say that function f has limit $L = \infty$ at improper point ∞ and write $\lim_{x \to \infty} f(x) = \infty$ if

$$\forall\, K > 0\ \exists\, x_1 > 0 \ \text{ such that } \forall\, x > x_1 \text{ is } f(x) > K\,.$$

(ii) We say that function f has limit $L = -\infty$ at improper point ∞ and write $\lim_{x \to \infty} f(x) = -\infty$ if

$$\forall\, L < 0\ \exists\, x_1 > 0 \ \text{ such that } \forall\, x > x_1 \text{ is } f(x) < L\,.$$

(iii) We say that function f has limit $L = \infty$ at improper point $-\infty$ and write $\lim_{x \to -\infty} f(x) = \infty$ if

$$\forall\, K > 0\ \exists\, x_2 < 0 \ \text{ such that } \forall\, x < x_2 \text{ is } f(x) > K\,.$$

(iv) We say that function f has limit $L = -\infty$ at improper point $-\infty$ and write $\lim_{x \to -\infty} f(x) = -\infty$ if

$$\forall\, L < 0\ \exists\, x_2 < 0 \ \text{ such that } \forall\, x < x_2 \text{ is } f(x) < L\,.$$

Theorem:

(i) Let $\lim_{x\to\pm\infty} f(x) = \infty$ and $\lim_{x\to\pm\infty} g(x) = \infty$ then

$$\lim_{x\to\pm\infty} f(x) + g(x) = \infty \qquad \lim_{x\to\pm\infty} f(x)\cdot g(x) = \infty\,.$$

(ii) Let $\lim_{x\to\pm\infty} f(x) = -\infty$ and $\lim_{x\to\pm\infty} g(x) = -\infty$ then

$$\lim_{x\to a} f(x) + g(x) = -\infty \qquad \lim_{x\to\pm\infty} f(x)\cdot g(x) = \infty\,.$$

(iii) Let $\lim_{x\to\pm\infty} f(x) = \infty$ and $\lim_{x\to\pm\infty} g(x) = -\infty$ then

$$\lim_{x\to a} f(x)\cdot g(x) = -\infty$$

(iv) Let $\lim_{x\to\pm\infty} f(x) = A \in \mathbb{R},\ A > 0$ and $\lim_{x\to\pm\infty} g(x) = \pm\infty$,

$$\lim_{x\to a} f(x)\cdot g(x) = \pm\infty\,.$$

(v) Let $\lim_{x\to\pm\infty} f(x) = A \in \mathbb{R}$ and $\lim_{x\to\pm\infty} g(x) = \pm\infty$ then

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0\,.$$

Improper Limits IV. (2)

Theorem: Let function f be bounded on some (x_0, ∞) or $(-\infty, x_0)$, respectively. Then

(i) if $\lim_{x\to\pm\infty} g(x) = \pm\infty$ then $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0$

(ii) if $\lim_{x\to\pm\infty} g(x) = 0$ then $\lim_{x\to\pm\infty} f(x)g(x) = 0$

Theorem: Let $\lim_{x\to\pm\infty} f(x) = A > 0$ and $\lim_{x\to\pm\infty} g(x) = 0$ then it holds:

(i) if $g(x) > 0$ on (a, ∞) or $(-\infty, a)$ then

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = +\infty$$

(ii) if $g(x) < 0$ on (a, ∞) or $(-\infty, a)$ then

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = -\infty$$

Definition: For all $n \in \mathbb{N}$ define $a_n \in \mathbb{R}$. We say that

$$a_1,\, a_2,\, a_3, \ldots$$

create a sequence of real numbers. Number a_n is called $n-$th term, n is index of number a_n. A sequence is briefly denoted $\{a_n\}_{n=1}^{\infty}$.

Remark: A sequence is a function defined on a subset of natural numbers $\mathbb{N}$ (or more generally integer numbers $\mathbb{Z}$):

$$a_n = f(n), \quad f : \mathbb{N} \to \mathbb{R}.$$

Arithmetic sequence:

$$a_n = a_1 + (n-1)d,$$

where $d \in \mathbb{R}$ is a common difference.

Geometric sequence:

$$a_n = a_1 \cdot q^n,$$

where $q \in \mathbb{R}$ is a common ratio (or quotient).

Limit of a sequence (2)

Definition: We say that a sequence $\{a_n\}_{n=1}^{\infty}$ has a limit

- $A \in \mathbb{R}$ if

$$\forall\, \mathcal{O}_\varepsilon(A)\ \exists\, n_0 \in \mathbb{N} \text{ such that } \forall\, n > n_0 \text{ is } a_n \in \mathcal{O}_\varepsilon(A)\,.$$

- $A = \pm\infty$ if

$$\forall\, K > 0\ \exists\, n_0 \in \mathbb{N} \text{ such that } \forall\, n \geq n_0 \text{ is } a_n > K\,,$$

$$\forall\, L < 0\ \exists\, n_0 \in \mathbb{N} \text{ such that } \forall\, n \geq n_0 \text{ is } a_n < L\,.$$

A sequence $\{a_n\}_{n=1}^{\infty}$ is called convergent, if it has a proper limit $A \in \mathbb{R}$.
A sequence $\{a_n\}_{n=1}^{\infty}$ is called divergent otherwise ($A = \pm\infty$ or does not exist).

Definition: Consider a sequence $\{a_n\}_{n=1}^{\infty}$

(i) If $\forall\, n \in \mathbb{N}$ is

$$a_n < a_{n+1} \quad (\text{resp. } a_n \leq a_{n+1})$$

we say that a sequence is increasing (non-decreasing, resp.).

(ii) If $\forall\, n \in \mathbb{N}$ is

$$a_n > a_{n+1} \quad (\text{resp. } a_n \geq a_{n+1})$$

we say that a sequence is decreasing (non-increasing, resp.).

Theorem:

- A decreasing or non-increasing sequence is bounded above.
- A increasing or non-decreasing sequence is bounded below.

Corollary: A monotone sequence has always limit. If it is bounded then the limit is proper.

One can prove that there exists a limit of the sequence

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}.$$

Definition: Denote

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}.$$

Number e is called Euler's number. This number is irrational.