Chapter 1: Real functions of one real variable

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{I}$, $\mathbb{R}$, $\mathbb{C}$

Definition: *A Cartesian product $M \times N$* of sets M and N is a coordinate system of ordered pairs (m, n) where $m \in M$ and $n \in N$.

$$M \times N = \{(m, n) | m \in M \wedge n \in N\}$$

Definition: Let $\emptyset \neq M \subset \mathbb{R}$. We say that M is *bounded above (bounded below)*, if there is a number $a \in \mathbb{R}$ such that for all $m \in M$ is $m \leq (\geq)\ a$. The number a is called *upper (lower) bound* of set M.

The set M is *bounded*, if it is simultaneously bounded below and above.

Definition: Let $\emptyset \neq M \subset \mathbb{R}$. We say that $\max M$ ($\min M$) is *maximum (minimum)* of set M, if:

(i) $\forall m \in M : m \leq \max M$ ($\geq \min M$)

(ii) $\max M$ ($\min M$) $\in M$

Definition: Let $M \subset \mathbb{R}$. A function f of a real variable is a rule which assigns to each $x \in M$ exactly one $y \in \mathbb{R}$.

Variable x is called *argument* or *independent variable* and variable y is called *dependent*.

The set M is called *the domain* of function f and denoted by $D(f)$. A set $\{y = f(x) | x \in D(f)\}$ is called *the range* of f and is denoted by $H(f)$.

- Notation: $y = f(x)$, $f : M \to \mathbb{R}$, $x \mapsto f(x)$, $x \mapsto y$
- General term: mapping

Definition *Graph of function* f is a set of ordered pairs of real numbers $(x, f(x))$, where $x \in D(f)$. We write

$$\text{graph } f = \{(x, f(x)) | x \in D(f)\}$$

Remark:

1. The domain is a part of definition of the function. If it is not given we consider the *natural domain*, that is the largest possible domain.
2. Two functions f and g are equal ($f = g$), if
   (i) $D(f) = D(g)$
   (ii) $\forall x \in D(f) : f(x) = g(x)$

Definition: Let f and g be real functions with domains $D(f)$ and $D(g)$.

- Let $H(f) \subseteq D(g)$. Then under the composition of function f and g we understand function h defined by

$$\forall x \in D(h) : h(x) = g(f(x))$$

  with $D(h) = D(f)$.
  Notation: $h = g \circ f$.

- If $H(f) \not\subseteq D(g)$, then by the domain of function $h = g \circ f$ we understand set

$$D(h) = \{x \in D(f) | f(x) \in D(g)\}$$

Remark: In general $g \circ f \neq f \circ g$.

Special classes of functions - injective functions

Definition: Function $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ is *injective (one to one)* on $M \subseteq D(f)$ if

$$\forall x_1, x_2 \in M : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

Remark:

- equivalently $\to$ proof that f is injective

$$\forall x_1, x_2 \in M : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

- negation $\to$ proof that f is not injective

$$\exists x_1, x_2 \in M : x_1 \neq x_2 \wedge f(x_1) = f(x_2)$$

Definition: Consider $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ and set $M \subseteq D(f)$. If for all $x_1, x_2 \in M$, $x_1 < x_2$ it holds

(i) $f(x_1) < f(x_2)$ is f *increasing* na M

(ii) $f(x_1) \leq f(x_2)$ is f *non-decreasing* na M

(iii) $f(x_1) > f(x_2)$ is f *decreasing* na M

(iv) $f(x_1) \geq f(x_2)$ is f *non-increasing* na M

If f satisfies any of condition $(i) - (iv)$ we call it *monotone*. If f has property (i) or (iii), we call it *strictly monotone*.

Proposition: A sum of two increasing (decreasing) functions is an increasing (decreasing) function.

Theorem: Let f be strictly monotone on set $M \subseteq \mathbb{R}$ then f is injective on M.

Definition:

- We say that function f is *bounded below* on its domain $D(f)$ if
$$\exists d \in \mathbb{R}\ \forall x \in D(f):\ d \leq f(x)$$
- We say that function f is *bounded above* on its domain $D(f)$ if
$$\exists h \in \mathbb{R}\ \forall x \in D(f):\ f(x) \leq h$$
- Function is *bounded* if it is bounded below and above.

Special classes of functions - even and odd functions

Definition:

- We say that function $f : D(f) \to \mathbb{R}$ is *even* if

$$\forall x \in D(f) : \; f(-x) = f(x)$$

- We say that function $f : D(f) \to \mathbb{R}$ is *odd* if

$$\forall x \in D(f) : \; f(-x) = -f(x)$$

Remark:

(i) Graph of an even function is symmetric with, respect to the y axis.

(ii) Graph of an odd function is symmetric with, respect to the origin.

(ii) **Domain of an even or odd function is always symmetric with respect to the origin!**

Definition: A function $f : D(f) \to \mathbb{R}$ is called *periodic* if $\exists p \in \mathbb{R}$, $p \neq 0$ such that:

(i) $x \in D(f) \Rightarrow x \pm p \in D(f)$

(ii) $\forall x \in D(f): \; f(x \pm p) = f(x)$

Number p is called a *period* of f. The smallest positive period is called *primitive*.

Theorem:

(i) If f is periodic with period p and function g such that $H(f) \subseteq D(g)$ then a composition $h(x) = g(f(x))$ is periodic with the same period p.

(ii) If f is periodic with period p and $a \in \mathbb{R}$, $a \neq 0$, then function $g(x) = f(ax)$ is periodic with period $\frac{p}{a}$.

Definition: Let $f : D(f) \to \mathbb{R}$ be an injective function with range $H(f)$. *Inverse function* of f (denoted f^{-1}) is defined by the relation

$$y = f(x) \Leftrightarrow x = f^{-1}(y)$$

Obviously the domain $D(f^{-1}) = H(f)$ and range $H(f^{-1}) = D(f)$
Remarks:

(i) Graph of f^{-1} is symmetric to the graph of f with respect to a line $y = x$.

(ii) $\forall x \in D(f) : f^{-1}(f(x)) = x$

(iii) $\forall y \in D(f^{-1}) = H(f) : f(f^{-1}(y)) = y$

(iv) $(f^{-1})^{-1} = f$

Exponential and logarithmic function

$$y = a^x \Leftrightarrow x = \log_a(y), x \in \mathbb{R}, y > 0, 1 \neq a > 0$$

Useful: $h(x) = f(x)^{g(x)} = e^{g(x)\ln(f(x))}$

Theorem:
Properties of functions $\arcsin(x)$, $\arccos(x)$, $\mathrm{arctg}(x)$, $\mathrm{arccotg}(x)$

$f(x)$	$\arcsin(x)$	$\arccos(x)$	$\mathrm{arctg}(x)$	$\mathrm{arccotg}(x)$
$D(f)$	$[-1, 1]$	$[-1, 1]$	$\mathbb{R}$	$\mathbb{R}$
$H(f)$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[0, \pi]$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(0, \pi)$
increasing	$\checkmark$	–	$\checkmark$	–
decreasing	–	$\checkmark$	–	$\checkmark$
even	–	–	–	–
odd	$\checkmark$	–	$\checkmark$	–
$f^{-1}(x)$	$\sin(x)$	$\cos(x)$	$\mathrm{tg}(x)$	$\mathrm{cotg}(x)$
	$x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$x \in [0, \pi]$	$x \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$x \in (0, \pi)$

Theorem: $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ for $x \in [-1, 1]$
$\mathrm{arctg}(x) + \mathrm{arccotg}(x) = \frac{\pi}{2}$ for $x \in \mathbb{R}$