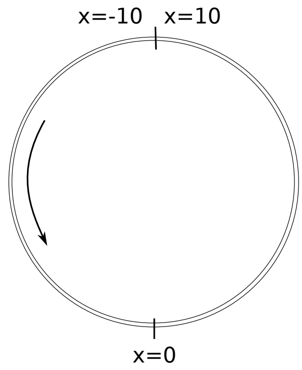
1. Consider a PDE describing the heat transfer in a thin copper ring of circumference 20cm

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where $T(x,t)$ is temperature, $t \in \langle 0, 5\,\text{minutes}\rangle$ and $x \in \langle -10\,\text{cm}, 10\,\text{cm}\rangle$ is position on the ring, the point $x = -10$ is indentical with the point $x = 10$. The termal diffusivity of the copper is $\alpha = 0.111\text{cm}^2/\text{s}$.



We prescribe periodic boundary conditions

$$T(-10,t) = T(10,t), \qquad \frac{\partial T}{\partial x}(-10,t) = \frac{\partial T}{\partial x}(10,t)$$

The initial temperature in the ring is prescribed by the function

$$T(x,0) = \sin\left(\frac{\pi x}{10}\right).$$

Solve this problem by the finite difference method:

(a) Derive the simple implicit formula for a general spatial step h and a general time step k.

(b) Solve it using $n = 40$ spatial steps and $m = 10$ time steps by any software.

(c) Plot the graph of the solution $T(x,t)$.

(d) At the time $t = 5$ minutes, compare the numerical solution with the exact solution

$$T(x,t) = \sin\left(\frac{\pi x}{10}\right) \exp\left(-\frac{\pi^2\,\alpha\,t}{100}\right)$$

2. Consider a PDE describing the heat transfer in a copper rod of lenght $L = 40$ cm.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where $T(x,t)$ is temperature, $t \in \langle 0, 20\,\text{minutes}\rangle$ and $x \in \langle 0, 40\,\text{cm}\rangle$. Thermal diffusivity of the copper is $\alpha = 0.111\text{cm}^2/\text{s}$.

Both endpoints of the rod are cooled to 0 degrees, thus

$$T(0,t) = 0, \qquad T(L,t) = 0$$

The initial temperature in the rod is prescribed by the function

$$T(x,0) = \sin\left(\frac{\pi x}{L}\right).$$

Solve this problem by the finite difference method:

(a)

(b) Derive the simple implicit formula for a general spatial step h and a general time step k.

(c) Solve it using $n = 80$ spatial steps and $m = 10$ time steps by any software.

(d) Plot the graph of the solution $T(x,t)$.

(e) At time $t = 10$ minutes and $t = 20$ minutes, compare the numerical solution with the exact solution

$$T(x,t) = \sin\left(\frac{\pi x}{L}\right) \exp\left(-\alpha \frac{\pi^2 t}{L^2}\right)$$

3. Consider a PDE describing the heat transfer in a copper rod of lenght $L = 40$ cm.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where $T(x,t)$ is temperature, $t \in \langle 0, 20\,\text{minutes}\rangle$ and $x \in \langle 0, 40\,\text{cm}\rangle$. Thermal diffusivity of the copper is $\alpha = 0.111\text{cm}^2/\text{s}$.

Both endpoints of the rod are perfectly insulated, thus

$$\frac{\partial T}{\partial x}(0,t) = 0, \qquad \frac{\partial T}{\partial x}(L,t) = 0$$

The initial temperature in the rod is prescribed by the function

$$T(x,0) = \cos(\frac{\pi x}{L}).$$

Solve this problem by the finite difference method:

(a)

(b) Derive the simple implicit formula for a general spatial step h and a general time step k.

(c) Solve it using $n = 80$ spatial steps and $m = 10$ time steps by any software.

(d) Plot the graph of the solution $T(x,t)$.

4. Consider a PDE describing the heat transfer in a copper rod of lenght $L = 40$ cm.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where $T(x,t)$ is temperature, $t \in \langle 0, 20\,\text{minutes}\rangle$ and $x \in \langle 0, 40\,\text{cm}\rangle$. Thermal diffusivity of the copper is $\alpha = 0.111\text{cm}^2/\text{s}$.

Both endpoints of the rod are cooled to 0 degrees, thus

$$T(0,t) = 0, \qquad T(L,t) = 0$$

The initial temperature in the rod is prescribed by the function

$$T(x,0) = \sin(\frac{\pi x}{L}).$$

Solve this problem by the finite difference method:

(a) Derive the explicit formula for a general spatial step h and a general time step k.

(b) Solve it using $n = 80$ spatial steps and $m = 1200$ time steps by any software.

(c) Plot the graph of the solution $T(x,t)$.

(d) Solve it again using $n = 80$ and $m = 1046$.

(e) Compare these two solutions at the time $t = 20$ minutes. Explain what happened.

5. Consider Black-Scholes model describing the price of the call options.

$$\frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC = 0$$

where $C(S,t)$ is the price of the option, S is the price of the underlying asset, t is time in years until maturity, $\sigma = 0.4$ is the volatility, $r = 0.1$ is the interest rate and $E = 10\$$ is the strike price.

At maturity ($t = 0$) the price of the option is given as

$$C(S,0) = \max(S - E, 0)$$

Boundary conditions

$$C(0,t) = 0, \qquad C(S_{\max}, t) = S_{\max} - Ee^{-rt}$$

where $S_{\max}$ is an artificial boundary, use $S_{\max} = 40$.

Solve this problem by the finite difference method:

(a) Derive the explicit formula for a general spatial step h and a general time step k.

(b) Solve it by any software using $n = 20$ steps in S and a suitable time step (stability!). Compute the solution to the time $t = 1$ year before maturity.

(c) Plot prices of the option $C(S,t)$, $S \in \langle 0, S_{\max} \rangle$ at times $t = 0.25, 0.5, 1$ year before maturity.

(d) Plot $C(S,t)$.

(e) What is the price of the option one year before maturity in case the price of the underlying asset is 12\$.

6. Consider Black-Scholes model describing the price of the call options.

$$\frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC = 0$$

where $C(S,t)$ is the price of the option, S is the price of the underlying asset, t is time in years until maturity, $\sigma = 0.4$ is the volatility, $r = 0.1$ is the interest rate and $E = 10\$$ is the strike price.

At maturity ($t = 0$) the price of the option is given as

$$C(S,0) = \max(S - E, 0)$$

Boundary conditions

$$C(0,t) = 0, \qquad C(S_{\max}, t) = S_{\max} - Ee^{-rt}$$

where $S_{\max}$ is an artificial boundary, use $S_{\max} = 40$.

Solve this problem by the finite difference method:

(a) Derive the simple implicit formula for a general spatial step h and a general time step k.

(b) Solve it by any software using $n = 160$ steps in S and $m = 16$ steps in time. Compute the solution to the time $t = 1$ year before maturity.

(c) Plot prices of the option $C(S,t)$, $S \in \langle 0, S_{\max} \rangle$ at times $t = 0.25, 0.5, 1$ year before maturity.

(d) Plot $C(S,t)$ as a 3D plot.

(e) What is the price of the option one year before maturity in case the price of the underlying asset is 12\$.

7. Consider Black-Scholes model describing the price of the call options.

$$\frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC = 0$$

where $C(S,t)$ is the price of the option, S is the price of the underlying asset, t is time in years until maturity, $\sigma = 0.4$ is the volatility, $r = 0.1$ is the interest rate and $E = 10\$$ is the strike price.

At maturity ($t = 0$) the price of the option is given as

$$C(S,0) = \max(E - S, 0)$$

Boundary conditions

$$C(0,t) = Ee^{-rt}, \qquad C(S_{\max}, t) = 0$$

where $S_{\max}$ is an artificial boundary, use $S_{\max} = 40$.

Solve this problem by the finite difference method:

(a) Derive the explicit formula for a general spatial step h and a general time step k.

(b) Solve it by any software using $n = 20$ steps in S and a suitable time step (stability!). Compute the solution to the time $t = 1$ year before maturity.

(c) Plot prices of the option $C(S,t)$, $S \in \langle 0, S_{\max}\rangle$ at times $t = 0.25, 0.5, 1$ year before maturity.

(d) Plot $C(S,t)$.

(e) What is the price of the option one year before maturity in case the price of the underlying asset is 8\$.

8. Consider Black-Scholes model describing the price of the call options.

$$\frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC = 0$$

where $C(S,t)$ is the price of the option, S is the price of the underlying asset, t is time in years until maturity, $\sigma = 0.4$ is the volatility, $r = 0.1$ is the interest rate and $E = 10\$$ is the strike price.

At maturity ($t = 0$) the price of the option is given as

$$C(S,0) = \max(E - S, 0)$$

Boundary conditions

$$C(0,t) = Ee^{-rt}, \qquad C(S_{\max}, t) = 0$$

where $S_{\max}$ is an artificial boundary, use $S_{\max} = 40$.

Solve this problem by the finite difference method:

(a) Derive the simple implicit formula for a general n and m.

(b) Solve it by any software using $n = 160$ steps in S and $m = 16$ steps in time. Compute the solution to the time $t = 1$ year before maturity.

(c) Plot prices of the option $C(S,t)$, $S \in \langle 0, S_{\max} \rangle$ at times $t = 0.25, 0.5, 1$ year before maturity.

(d) Plot $C(S,t)$.

(e) What is the price of the option one year before maturity in case the price of the underlying asset is 8\$.