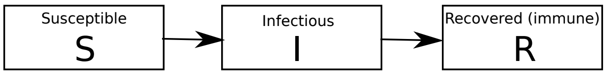
1. Consider an SIR model for spreading diseases described by the system of ordinary differential equations



$$\begin{aligned} S' &= -\beta S I \\ I' &= \beta S I - \nu I \\ R' &= \nu I \end{aligned}$$

where $N = S + I + R$ is the whole population. Number of recovered R can be computed $R = N - S - I$ therefore it is sufficient to solve just two ODEs.

We consider a closed population $N = 5000$, an island where a ship arrived. Ten days after the arrival there are 500 infected. We know that at the beginning $R(0) = 0$, thus $S(0) + I(0) = N$, and we want to know how many infected arrived on the ship.

a) Use the shooting method to solve the problem on time interval $t = \langle 0, 10 \rangle$ and determine the value $I(0)$.

b) Using this initial condition, solve the system as an initial value problem. When is the epidemic over ($I(t) < 0.5$)? Plot the time evolution of the epidemic (numbers S,I and R in time).

Use the following parameters $\nu = 0.25$ (a patient is infectious for 4 days), $R_0 = 3$ (corresponds to smallpox), where $R_0 = \frac{\beta}{\nu} N$ is the reproduction number.

2. Height y of a ball thrown up in the air is described by the ODE

$$my'' = \pm F_D - mg$$

where the drag force $F_D = \frac{1}{2} C_p A \rho v^2$ acts against the direction of motion. Here v is the velocity of the ball, $C_p = 0.47$ is the drag coefficient (for sphere in the air), $\rho = 1.225\mathrm{kg/m}^3$ is the air density, $A = \pi r^2$ the cross sectional area of the ball with radius $r = 0.1\mathrm{m}$, $m = 0.4\mathrm{kg}$ is the mass of the ball and $g = 9.81\mathrm{m/s}^2$ is the gravitational acceleration. The ball is thrown up in the air, and after 5s we want the ball to be 40m above ground. At what velocity do we need to throw the ball?

   Use the finite difference method.

3. Height y of a ball thrown up in the air is described by the ODE

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Use the shooting method.

4. Fall of a distant object to Earth is described by the ODE

$$r'' = -\frac{gR^2}{r^2}$$

where r is the distance of the object from **Earth center**, $g = 9.81\mathrm{m/s}^2$ is the gravitational acceleration and $R = 6371\mathrm{km}$ is the Earth radius. At time $t = 0$ there was an explosion on a geostationary satelite (35887km above **Earth surface**), debris fell on Earth after 2 hours.

Use the finite difference method to model the satelite crash. Plot graphs of satelite velocity and its distance from Earth center in time.

(We assume, that falling debris doesn't change mass. We neglect air resistance.)

5. Consider a chemical reaction

   $\mathrm{A} + 2\,\mathrm{B} \xrightarrow{\mathrm{k}} 3\,\mathrm{C}$

   which can be written as a system of differential equations

$$
\begin{aligned}
C_A' &= -kC_AC_B^2 \\
C_B' &= -2kC_AC_B^2 \\
C_C' &= 3kC_AC_B^2,
\end{aligned}
$$

   where C_A, C_B, C_C are concentrations of substances A, B, C and $k = 0.1$ is the reaction rate.

   Assume that we know the concentration C_A at the beginning of the reaction $C_A(0) = 2$. At time $t = 10$ we want $C_C(10) = 4$.

   By the shooting method find the initial concentration of B. Plot the time evolution of concentrations of A, B, C during the reaction.

6. Consider steady-state heat transfer in a thin steel rod of legnth 1 m.

$$\alpha \frac{\partial^2 T}{\partial x^2} = \mu(T - T_a)$$

where $T(x), x \in \langle 0, 1 \rangle$ is temperature, T_a is temperature of the surrounding air and α is a heat coefficient describing the diffusion of heat into surroundings.

Consider fixed temperature at the ends of the rod $T(0) = 25°C$ and $T(1) = 75°C$. Temperature of surroundings $T_a = 20°C$ and heat coefficients $\alpha = 1.17 \times 10^{-5} \mathrm{m}^2/\mathrm{s}$ and $\mu = 1.45 \times 10^{-4}$.

Compute the solution analytically and then numerically using shooting method. Compare solutions.

7. Consider steady-state heat transfer in a thin steel rod of legnth 1 m.

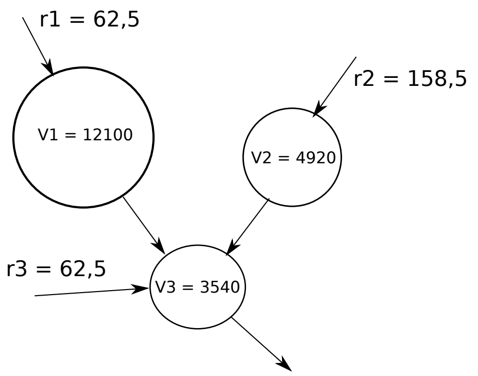
$$\alpha \frac{\partial^2 T}{\partial x^2} = \mu(T - T_a)$$

where $T(x), x \in \langle 0, 1\rangle$ is temperature, T_a is temperature of the surrounding air and α and μ are heat coefficients describing the diffusion of heat into surroundings.

Consider left end of the rod to be heat insulated, thus $T'(0) = 0$ and consider a fixed temperature at the right end, $T(1) = 75°C$. Temperature of surroundings $T_a = 20°C$ and heat coefficients $\alpha = 1.17 \times 10^{-5}\mathrm{m}^2/\mathrm{s}$ and $\mu = 1.45 \times 10^{-4}$.

Solve using finite difference method. What temperature is at the left end of the rod? Use Richardson extrapolation to estimate error of this result.

8. Consider a model describing the cleaning of the water in 3 connected lakes (see figure). Assume that only clean water flows into the conta-



minated lakes and the volume of the lakes is constant and the contaminants are water-soluble. The model is described by the following system of differential equations

$$
\begin{aligned}
x_1' &= -\frac{r_1}{V_1}x_1 \\
x_2' &= -\frac{r_2}{V_2}x_2 \\
x_3' &= \frac{r_1}{V_1}x_1 + \frac{r_2}{V_2}x_2 - \frac{(r_1 + r_2 + r_3)}{V_3}x_3
\end{aligned}
$$

where x_1, x_2, x_3 are concentrations of contaminants [tun/km^3], V_1, V_2, V_3 are volumes of the lakes [km^3] and r_1, r_2, r_3 are inflows per year [km^3/rok]. At the time $t = 0$ the contamination of the upper lakes is known $x_1(0) = 5$ tun/km^3, $x_2(0) = 10$ tun/km^3 and we know that the concentration of contaminants at the third lake decreases by 5% in two years. Use the shooting method to find the initial contamination of the third lake. Using that, compute and plot the concentrations in all three lakes in the next years. In how many year the concentration x_3 decreases below 5% of the original concentration?

9. Consider chemical reactions

   $\mathrm{A} + \mathrm{B} \xrightarrow{\mathrm{k_1}} 2\,\mathrm{B}$

   $\mathrm{B} \xrightarrow{\mathrm{k_2}} \mathrm{C}$

   which can be written in the form of differential equations

$$\begin{aligned} C_A' &= -k_1 C_A C_B \\ C_B' &= k_1 C_A C_B - k_2 C_B \\ C_C' &= k_2 C_B, \end{aligned}$$

   where C_A, C_B, C_C are concentrations of A, B, C and $k_1 = 0.15\,\mathrm{l\,mol^{-1}s^{-1}}$, $k_2 = 0.05\ \mathrm{s}^{-1}$ are reaction rates.

   Assume that the concetnration C_A is known at the beginning of the process and we know the desired concentration of C at $t = 20$ s. Thus

$$\begin{aligned} C_A(0) &= 0.2\ \mathrm{mol/l} \\ C_C(0) &= 0\ \mathrm{mol/l} \\ C_C(20) &= 0.3\ \mathrm{mol/l}. \end{aligned}$$

   Use the shooting method to find the concentrations of A, B, C during the reaction $t \in \langle 0, 20 \rangle$. What is the necessary initial concentration of B?