1 Differential equations

Differential equation is an equation which relates a function $y(x)$ with its derivatives $y'(x), y''(x), y'''(x), \dots$ and the independent variable x, e.g.

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \tag{1}$$

where F is a function in $n+2$ indeterminates.

Definition 1 *By a* solution *to a differential equation (1) we refer to a function $y(x)$ defined on an interval I which satisfies (1) for all $x \in I$.*
The general solution *to (1) is a collection of all solutions to (1). One specific solution to (1) is called a* particular solution. *The graph of a particular solution is called the* integral curve.

When searching for a particular solution to a differential equation we usually deal with two problems:

1. **Initial value problem:**

$$\begin{aligned} &F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \\ &y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1} \end{aligned}$$

   - find a particular solution $y_P(x)$, $x \in I$ to the differential equation $F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$ such that it satisfies the *initial conditions* $y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$, i.e. such that

$$y_P(x_0) = y_0, y_P'(x_0) = y_1, \dots, y_P^{(n-1)}(x_0) = y_{n-1}.$$

   Note that $x_0 \in I$.

2. **Boundary value problem**

$$\begin{aligned} &F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \\ &y(x_0) = y_0, y(x_1) = y_1 \end{aligned}$$

   - find a particular solution $y_P(x)$, $x \in I$ to the differential equation $F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$ such that it satisfies the *boundary conditions* $y(x_0) = y_0, y(x_1) = y_1$, i.e. such that

$$y_P(x_0) = y_0, y_P(x_1) = y_1.$$

   Note that $[x_0, x_1] \subseteq I$ (for $x_0 < x_1$).

Definition 2 Order *of a differential equation $F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$ is n - the highest order of the derivative of $y(x)$ appearing in the equation.*

Definition 3 *A* linear differential equation *of order n is a differential equation of order n which can be written in the form*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = b(x),$$

where $b(x), a_i(x)$, $i = 0, \dots, n$ are continuous functions on an interval I and $a_0(x) \neq 0$ for all $x \in I$.
Differential equations which are not linear are called nonlinear.

1.1 Separable differential equations

A first order differential equation $F(x, y(x), y'(x)) = 0$ is called *separable* if there exist functions f and g such that

$$y'(x) = f(x)\,g(y). \tag{2}$$

Theorem 1 ***(Existence and uniqueness of solutions)***
Consider a differential equation (2). If $f(x)$ is a continuous function on an open interval (a, b) and $g(y)$ is a continuously differentiable function on an open interval (c, d), then for every point of the rectangle $O = (a, b) \times (c, d)$ there is exactly one integral curve passing through it. In other words, there exists a unique solution to (2) satisfying an initial condition $y(x_0) = y_0$, where $(x_0, y_0) \in O$.

Note that the line with the direction $f(x_0)g(y_0)$ passing through a point (x_0,y_0) is the tangent line to the integral curve corresponding to the particular solution of the initial value problem $y' = f(x)\,g(y)$, $y(x_0) = y_0$.
If a short line segment of direction $f(x)g(y)$ is drawn at each point (x,y) of the rectangle O (i.e. it is a line segment of the tangent line to the integral curve, all passing through the point (x,y)), one obtains so-called *slope* or *direction field* for the equation $y' = f(x)\,g(y)$.

Theorem 2 ***(Separation of variables)*** *Let f be a continuous function on an interval (a,b) and let g be a continuously differentiable function on an interval (c,d). Then the following holds.*

(i) If $g(y_0) = 0$ for some $y_0 \in (c,d)$, then the constant function

$$y(x) \equiv y_0,\ x \in (a,b)$$

is a solution to $y' = f(x)\,g(y)$.

(ii) If $g(y) \neq 0$ for all $y \in (c,d)$, then the general solution to $y' = f(x)\,g(y)$ on the rectangle $(a,b)\times(c,d)$ is of the form

$$y(x) = G^{-1}(F(x)+C),$$

where $F(x) = \int f(x)\mathrm{d}x$ and $G(y) = \int \frac{1}{g(y)}\mathrm{d}y$.

The proof of the theorem provides us with the algorithm for solving separable differential equations.

Algorithm 1 *Consider the differential equation (2) such that $f(x)$ is continuous on (a,b) and $g'(y)$ is continuous on (c,d).*

1. *Determine all points y_0 such that $g(y_0) = 0$.*
   Then $y(x) = y_0$, $x \in (a,b)$ is a constant solution to (2).
2. *Note that $y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$ and thus $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\,g(y)$, $x \in (a,b)$, $y \in J \subseteq (c,d)$, where J is an interval which does not contain y_0.*
3. *Separate the variables:*
$$\frac{\mathrm{d}y}{g(y)} = f(x)\mathrm{d}x$$
4. *Integrate both sides, the left side w.r.t. y and the right w.r.t. x,*
$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\mathrm{d}x$$
5. *Let $G(y)$ be an antiderivative of $\frac{1}{g(y)}$ and let $F(x)$ be an antiderivative of $f(x)$. Then*
$$y(x) = G^{-1}(F(x)+C),\ C \in \mathbb{R},\ x \in (a,b)$$
   is the general solution (together with the constant solution $y(x) = y_0, x \in (a,b)$) to (2).

1.2 Linear differential equations of order 1

Definition 4 *Let $a_0(x)$, $a_1(x)$, $b_1(x)$, $a(x)$, $b(x)$ be continuous functions on an open interval I. If $\forall x \in I : a_0(x) \neq 0$, then the equation*

$$a_0(x)y' + a_1(x)y = b_1(x) \ \text{ or equivalently } \ y' + a(x)y = b(x)$$

is a first order linear differential equation.

Further, if $\forall x \in I : b(x) = 0$, the equation $y' + a(x)y = 0$ is said to be homogeneous *first order linear differential equation (HLDE). Otherwise, if $\exists x \in I : b(x) \neq 0$, then the equation $y' + a(x)y = b(x)$ is called* nonhomogeneous *first order linear differential equation (NLDE).*

Theorem 3 ***(general solution to HLDE of order*** 1***)***
A collection of all solutions to a first order HLDE

$$y' + a(x)y = 0 \tag{3}$$

is of the form

$$y_H(x) = C\mathrm{e}^{-A(x)},\ C \in \mathbb{R},\ \textit{where } A(x) = \int a(x)\mathrm{d}x.$$

Theorem 4 ***(general solution to NLDE of order*** 1***)***
The general solution to a first order NLDE

$$y' + a(x)y = b(x) \tag{4}$$

is of the form

$$y = y_P + y_H,$$

where y_P is a particular solution to (4) and y_H is the general solution to the corresponding HLDE, i.e. to (3).

Theorem 5 ***(variation of constant)***
Let $y_H(x) = C\varphi(x)$ be the general solution to (3). If a function $c(x)$ satisfies the equation

$$c'(x)\varphi(x) = b(x),$$

then the function

$$y_p(x) = c(x)\varphi(x)$$

is a particular solution to (4).

Note that the theorem above can be formulated as:
Consider a NLDE $a_0(x)y' + a_1(x)y = b_1(x)$ such that $a_0(x) \neq 0$ for all x in an interval I. Let $y_H(x) = C\varphi(x)$ be the general solution to the corresponding HLDE $a_0(x)y' + a_1(x)y = 0$. If a function $c(x)$ satisfies the equation

$$c'(x)\varphi(x) = \frac{b_1(x)}{a_0(x)},$$

then $y_P(x) = c(x)\varphi(x)$ is a particular solution to $a_0(x)y' + a_1(x)y = b_1(x)$.

Algorithm 2 *Consider (4) on an interval I, i.e. $x \in I$.*

1. *Find the general solution to (3):*

   $$y_H(x) = C\mathrm{e}^{-A(x)},\ C \in \mathbb{R},\ A(x) = \int a(x)\mathrm{d}x.$$

   Denote $\varphi(x) = \mathrm{e}^{-A(x)}$, i.e. $y_H(x) = C\,\varphi(x)$.

2. *Find a particular solution to (4) (by the variation of the parameter):*
   Assume $y_P(x) = c(x)\varphi(x)$, where $c(x)$ is a function defined on I.

   (i) Substitute for y_P in (4):
   $$\begin{aligned} c'(x)\varphi(x) + c(x)\varphi'(x) + a(x)c(x)\varphi(x) &= b(x) \\ c'(x)\varphi(x) &= b(x) \end{aligned}$$

   (ii) $c(x) = \int \frac{b(x)}{\varphi(x)}\mathrm{d}x$

3. *The general solution to (4) is: $y(x) = y_P(x) + y_H(x),\ \ x \in I$*

1.3 Euler method

The Euler method is a numerical procedure for solving ordinary differential equations with a given initial value. It is the most basic explicit method for numerical integration of ordinary differential equations.

Consider the initial value problem

$$\begin{aligned} y' &= f(x,y), \;\; x \in [a,b], \\ y(a) &= y_0. \end{aligned}$$

The steps of the Euler method to approximate the particular solution to the initial value problem above are as follows:

1. Divide the interval $[a,b]$ into n subintervals with a chosen division step h, i.e. $n = \frac{b-a}{n}$ and the division

   $$a = x_0 < x_1 < \cdots < x_n = b \;\text{ is equidistant with }\; x_i = x_{i-1} + h, i = 1, \ldots, n.$$

2. Compute the approximations y_i of the values $y(x_i)$ of the particular solution $y(x)$ to the initial value problem at the division points x_i, $i = 0, \ldots, n$ as

   $$y_{i+1} = y_i + h\,f(x_i, y_i), \quad i = 0, \ldots, n-1.$$

3. The piecewise-linear function the graph of which is joining the points (x_i, y_i), $i = 0, \ldots, n$ is the approximation of the particular solution to the initial value problem by the Euler method with the step h.

Because the global error $E(h) = y_n - y(b)$ is proportional to h, we say the Euler method is a numerical method of the first order.