1 Indefinite integrals

Definition 1 *Let f be a function defined on an interval I. Function F defined on I such that*

$$\forall x \in I: \quad F'(x) = f(x)$$

is called an antiderivative *(other used names: primitive, primitive function) of f on I.*

Because constant functions $c(x) = c \in \mathbb{R}$ have zero derivatives, for every antiderivative F of f on I it holds that

$$f = F' = F' + 0 = F' + c' = (F + c)'.$$

Hence $F + c$, where $c \in \mathbb{R}$ are antiderivatives of f on I. So, antiderivatives are unique only up to a constant.

Definition 2 *Let f be a function defined on an interval I. A set of all antiderivatives of f on I is called* indefinite integral *of f on I and we denote it*

$$\int f(x)\,\mathrm{d}x = \{F(x) + c \,|\, c \in \mathbb{R},\ F \text{ is an antiderivative of } f \text{ on } I\}.$$

Theorem 1 ***(Existence of antiderivatives)***
Let f be a continuous function on I. Then f has an antiderivative on I.

According to the theorem above every continuous function has an antiderivative. However, it might not always be possible to find its explicit formula. Examples of such functions are the error function $\int \mathrm{e}^{-x^2}\,\mathrm{d}x$, Fresnel function $\int \sin(x^2)\,\mathrm{d}x$, trigonometric integral function $\int \frac{\sin x}{x}\,\mathrm{d}x$ and logarithmic integral function $\int \frac{1}{\ln(x)}\,\mathrm{d}x$.

Antiderivatives of basic functions		
$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1},\ n \in \mathbb{R},\ n \neq -1$	$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a},\ a > 0,\ a \neq 1$	$\int \frac{1}{x}\,\mathrm{d}x = \ln\|x\|$
$\int \sin(x)\,\mathrm{d}x = -\cos(x)$	$\int \frac{1}{\cos^2(x)}\,\mathrm{d}x = \tan(x)$	$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(x)$
$\int \cos(x)\,\mathrm{d}x = \sin(x)$	$\int \frac{1}{\sin^2(x)}\,\mathrm{d}x = -\cot(x)$	$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin(x)$
$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln(\|f(x)\|)$	$\int \frac{1}{\sqrt{x^2+a}}\,\mathrm{d}x = \ln(\|x + \sqrt{x^2+a}\|),\ a \neq 0$	

Theorem 2 ***(Properties of antiderivatives)***
The following holds:

(i) $\int k f(x)\,\mathrm{d}x = k \int f(x)\,\mathrm{d}x$, *where $k \in \mathbb{R}$ is a constant*

(ii) $\int (f(x) \pm g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$

Methods for computing antiderivatives:

- Integration by parts (per-partes)
- Integration by change of variable (substitution method)
- Integration of rational functions (partial fraction decomposition)

1.1 Per-partes method

Theorem 3 *Suppose functions u and v have continuous derivatives on an interval I. Then*

$$\int u(x)\,v'(x)\,\mathrm{d}x = u(x)\,v(x) - \int u'(x)\,v(x)\,\mathrm{d}x \quad \text{on } I.$$

1.2 Substitution method

Theorem 4 *Let $f(t)$ be a function continuous on an interval (a,b). Let $\varphi(x)$ be a continuously differentiable function on an interval (α,β) such that it maps the interval (α,β) onto the interval (a,b).*

(i) Then

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t = F(t) = F(\varphi(x)),$$

where F is an antiderivative of f on (a,b).

(ii) Further suppose $\forall x \in (\alpha,\beta): \ \varphi'(x) \neq 0$. Then

$$\int f(t)\,\mathrm{d}t = \int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = F(x) = F(\varphi^{-1}(t)),$$

where F is an antiderivative of $f(\varphi(x))\varphi'(x)$ on (α,β).

In both cases $t = \varphi(x)$ is the used substitution.

1.3 Integration by partial fractions

1.3.1 Polynomial factorization

Let us recall that $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree $n \in \mathbb{N}$ with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$. Specifically, $P_0(x) = a_0$ is a polynomial of degree zero. A number $\alpha \in \mathbb{C}$ such that $P_n(\alpha) = 0$ is called a *root* of the polynomial $P_n(x)$.

Proposition 1 *Let $\alpha \in \mathbb{C}$ be a root of a polynomial $P_n(x)$ of degree $n \in \mathbb{N}$. Then*

$$P_n(x) = (x-\alpha)Q(x)\,,$$

where $Q(x)$ is a polynomial of degree $(n-1)$.

Definition 3 *We say that a root $\alpha \in \mathbb{C}$ of a polynomial $P(x)$ is of* multiplicity *$k \in \mathbb{N}$ if there exists a polynomial $Q(x)$ such that*

$$P(x) = (x-\alpha)^k Q(x) \text{ and } Q(\alpha) \neq 0.$$

In case $k = 1$, we refer to α as a simple root *of $P(x)$. If $k \geq 2$, α is referred to as a* multiple *or* repeated root *of $P(x)$.*

Proposition 2 *$\alpha \in \mathbb{C}$ is a root of multiplicity $k \in \mathbb{N}$ of a polynomial $P(x)$ if and only if*

$$P(\alpha) = 0,\ P'(\alpha) = 0, \ldots, P^{(k-1)}(\alpha) = 0 \text{ and } P^{(k)}(\alpha) \neq 0.$$

Theorem 5 *Every non-zero polynomial of degree n has, counted with multiplicity, exactly n roots.*

A non-constant polynomial is called *irreducible* if it cannot be factored into the product of two non-constant polynomials. In case of univariate polynomials (i.e. polynomials of one variable) with real coefficients, irreducible polynomials are either linear, i.e. of the form $a_1 x + a_0$ or quadratic with complex conjugate roots, i.e. of the form $a_2 x^2 + a_1 x + a_0$ with $D = a_1^2 - 4a_2 a_0 < 0$.

Then by a *polynomial factorization* we mean the process of expressing a polynomial as the product of irreducible factors. This decomposition is unique up to the order of the factors and the multiplication of the factors by non-zero constants whose product is 1.

1.3.2 Rational functions

Definition 4 *A function* $f(x) = \dfrac{P(x)}{Q(x)}$*, where* $P(x)$ *and* $Q(x)$ *are polynomials, is called a* rational function. *If the degree of* $P(x)$ *is smaller than the degree of* $Q(x)$*, the rational function* f *is said to be* proper. *It is said to be* improper *otherwise.*

Proposition 3 *Every rational function can be written as a sum of a polynomial and a proper rational function.*

By a *partial fraction decomposition* we mean the operation that consists in expressing a rational function as a sum of a polynomial (possibly zero) and one or several fractions with simpler denominators.

Consider a rational function $f(x) = \dfrac{P(x)}{Q(x)}$. By long division of $P(x)$ and $Q(x)$ and by polynomial factorization of $Q(x)$ one derives

$$f(x) = p_1(x) + \frac{p_2(x)}{Q(x)} = p_1(x) + \frac{p_2(x)}{C(x-a_1)^{j_1}\cdots(x-a_m)^{j_m}(x^2+b_1x+c_1)^{k_1}\cdots(x^2+b_nx+c_n)^{k_n}},$$

where

(i) $p_1(x), p_2(x)$ are polynomials such that the degree of $p_2(x)$ is strictly smaller than the degree of $Q(x)$,

(ii) $C, a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{R}$ with $b_i^2 - 4c_i < 0$,

(iii) the terms $(x-a_i)$ are linear factors of $Q(x)$ which correspond to the real roots of $Q(x)$, and the terms $(x^2+b_ix+c_i)$ are irreducible quadratic factors of $Q(x)$ which correspond to the pairs of complex conjugate roots of $Q(x)$,

(iv) $j_1, \ldots, j_m, k_1, \ldots, k_n \in \mathbb{N}$ correspond to the multiplicities of the respective roots of $Q(x)$.

Then the partial fraction decomposition of $f(x)$ is the following:

$$f(x) = p_1(x) + \frac{1}{C}\left(\sum_{i=1}^{m}\sum_{r=1}^{j_i}\frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}\frac{B_{ir}x+C_{ir}}{(x^2+b_ix+c_i)^r}\right),$$

where the A_{ir}, B_{ir}, and C_{ir} are real constants. There are a number of ways these constants can be found.

1.3.3 Integration by partial fractions

Consider a rational function $f(x)$ with the partial fraction decomposition of the form

$$f(x) = p_1(x) + \sum_{i=1}^{m}\sum_{r=1}^{j_i}\frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n}\frac{B_ix+C_i}{(x^2+b_ix+c_i)},$$

where $p_1(x)$ is a polynomial, $A_{ir}, a_i, B_i, b_i, C_i, c_i \in \mathbb{R}$ and $m, j_i, n \in \mathbb{N}$. Then the indefinite integral of $f(x)$ can be derived as

$$\int f(x)\,\mathrm{d}x = \int p_1(x)\,\mathrm{d}x + \sum_{i=1}^{m}\sum_{r=1}^{j_i}\int\frac{A_{ir}}{(x-a_i)^r}\,\mathrm{d}x + \sum_{i=1}^{n}\int\frac{B_ix+C_i}{(x^2+b_ix+c_i)}\,\mathrm{d}x$$

with

(i) $\int \frac{A_{ir}}{x-a_i}\,\mathrm{d}x = A_{ir}\ln(|x-a_i|)$

(ii) $\int \frac{A_{ir}}{(x-a_i)^r}\,\mathrm{d}x = \frac{A_{ir}}{(1-r)(x-a_i)^{r-1}}, \quad r = 2, 3, \ldots$

(iii) $\int \frac{B_ix+C_i}{(x^2+b_ix+c_i)}\,\mathrm{d}x = \frac{B_i}{2}\ln(|x^2+b_ix+c_i|) + \frac{C_i - \frac{b_iB_i}{2}}{\sqrt{c_i - \left(\frac{b_i}{2}\right)^2}}\arctan\left(\frac{x+\frac{b_i}{2}}{\sqrt{c_i-\left(\frac{b_i}{2}\right)^2}}\right)$

2 Definite integrals

Given a function f of a real variable x and an interval $[a,b]$ of the real line, the definite integral $\int_a^b f(x)\,\mathrm{d}x$ is defined informally as the signed area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x=a$ and $x=b$. The area above the x-axis adds to the total and that below the x-axis subtracts from the total.

2.1 Riemann definition of definite integral

By a *division* of an interval $[a,b]$ we mean a finite set

$$D:\ a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=b$$

of points decomposing the interval $[a,b]$ into finitely many subintervals $[x_{i-1},x_i]$, $i=1,\ldots,n$. The points $x_0,x_1,\ldots,x_n$ are called *division points*. We say that a division D is *equidistant* if all subintervals $[x_{i-1},x_i]$, $i=1,\ldots,n$ are of the same length. Hence,

$$x_i=a+i\frac{b-a}{n},\ i=0,1,\ldots,n, \qquad x_i-x_{i-1}=\frac{b-a}{n},\ i=1,\ldots,n.$$

Definition 5 *Let f be a continuous function on an interval $[a,b]$. Given an equidistant division $D:\ a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=b$ of $[a,b]$, let $\{c_i\}_{i=1}^n$ be a sequence of real numbers such that $c_i\in[x_{i-1},x_i]$, $i=1,\ldots,n$ is arbitrarily chosen. The sum*

$$S_n(f)=\sum_{i=1}^n f(c_i)(x_i-x_{i-1})=f(c_1)(x_1-x_0)+\cdots+f(c_n)(x_n-x_{n-1})$$

is called the Riemann sum *of the function f corresponding to the division D and the points $\{c_i\}_{i=1}^n$.*

Theorem 6 *Let f be a continuous function on an interval $[a,b]$. Then the limit $\lim_{n\to\infty} S_n(f)$ exists and its value is independent on the choice of the points $c_i\in[x_{i-1},x_i]$, $i=1,\ldots,n$.*

Definition 6 *Let f be a continuous function on an interval $[a,b]$. The value $\lim_{n\to\infty} S_n(f)$ is called the* Riemann integral of f over $[a,b]$ *and it is denoted $(\mathcal{R})\int_a^b f(x)\,\mathrm{d}x$. Hence,*

$$(\mathcal{R})\int_a^b f(x)\,\mathrm{d}x=\lim_{n\to\infty} S_n(f).$$

The points a and b are called the limits of the integral, *a is the* lower *limit and b is the* upper *limit.*

2.2 Newton definition of definite integral

Definition 7 *Let f be a function defined on an interval I and let F be its antiderivative on I. Let $a,b\in I$. The* Newton integral of f over the interval $[a,b]$ (from a to b) *is the real number $F(b)-F(a)$. We write*

$$(\mathcal{N})\int_a^b f(x)\,\mathrm{d}x=[F(x)]_a^b=F(b)-F(a).$$

Remark 1

- *The value of the Newton integral of f does not depend on the choice of an antiderivative of f.*
- *If f is a continuous function on an interval I and $x_0\in I$, then*

  $$G(x)=\int_{x_0}^x f(t)\,\mathrm{d}t,\quad x\in I.$$

  is an antiderivative of f on I such that $G(x_0)=0$.

Theorem 7 *Let f be a continuous function on an interval $[a,b]$. Then f is both Newton and Riemann integrable and the values of the respective integrals are the same, i.e*

$$(\mathcal{N})\int_a^b f(x)\,\mathrm{d}x = (\mathcal{R})\int_a^b f(x)\,\mathrm{d}x.$$

Because during this course we only deal with continuous functions while studying integral calculus, the theorem above allows us not to make the distinction between Newton and Riemann definition of definite integrals. Thus we will use the notation

$$\int_a^b f(x)\,\mathrm{d}x = (\mathcal{N})\int_a^b f(x)\,\mathrm{d}x = (\mathcal{R})\int_a^b f(x)\,\mathrm{d}x.$$

2.2.1 Evaluating definite integrals

Theorem 8 *The following holds:*

(i) $\forall k \in \mathbb{R}: \int_a^b k f(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$

(ii) $\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$

(iii) $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x, \quad c \in (a,b)$

(iv) $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

Theorem 9 ***(Per partes method)***
Suppose functions u and v have continuous derivatives on an interval $[a,b]$. Then

$$\int_a^b u(x)v'(x)\,\mathrm{d}x = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,\mathrm{d}x.$$

Theorem 10 ***(Substitution method)***
Let $f(t)$ be a function continuous on an interval $[a,b]$. Let $\varphi(x)$ be a continuously differentiable function on an interval $[\alpha,\beta]$ such that it maps the interval $[\alpha,\beta]$ onto the interval $[a,b]$ and $\varphi(\alpha) = a$, $\varphi(\beta) = b$.

(i) Then

$$\int_\alpha^\beta f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = [F(t)]_a^b,$$

where F is an antiderivative of f on $[a,b]$.

(ii) Further suppose $\forall x \in [\alpha,\beta]: \ \varphi'(x) \neq 0$. Then

$$\int_a^b f(t)\,\mathrm{d}t = \int_\alpha^\beta f(\varphi(x))\varphi'(x)\,\mathrm{d}x = [F(x)]_\alpha^\beta,$$

where F is an antiderivative of $f(\varphi(x))\varphi'(x)$ on $[\alpha,\beta]$.

In both cases $t = \varphi(x)$ is the used substitution.

2.3 Improper integrals

Definition 8 *When either the integrand or the integration domain of a definite integral are unbounded, the resulting integral is called* improper.

Definition 9 *Let f be a continuous function defined on an open interval (a,b), let F be an antiderivative of f on (a,b) and let at least one of the following conditions be satisfied:*

(i) $a = -\infty$,

(ii) $b = +\infty$,

(iii) f *is unbounded on* (a,b).

Then $\int_a^b f(x)\,\mathrm{d}x$ *is improper and*

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = \lim_{x\to b-} F(x) \; - \; \lim_{x\to a+} F(x)$$

if the right-hand side of the equation is well defined. We say the improper integral converges *if the right-hand side is a finite number, it* diverges *otherwise.*

Proposition 4 *Let a function* f *be continuous on* $(a,b) \subseteq \mathbb{R}$ *except finitely many points* $c_1, \ldots, c_k$*. Then*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^{c_1} f(x)\,\mathrm{d}x + \int_{c_1}^{c_2} f(x)\,\mathrm{d}x + \cdots + \int_{c_k}^b f(x)\,\mathrm{d}x.$$

2.4 Applications of definite integrals

2.4.1 Computing areas of plane figures

Theorem 11 *Let* f *be a continuous nonnegative function defined on an interval* $[a,b]$*. Then the area* P *of the planar figure bounded by the x-axis, by the graph of* f *and by the vertical lines* $x = a$*,* $x = b$ *equals*

$$P = \int_a^b f(x)\,\mathrm{d}x\,.$$

Corollary 1 *Let* f *be a continuous negative function defined on an interval* $[a,b]$*. Then the area* P *of the planar figure bounded by the x-axis, by the graph of* f *and by the vertical lines* $x = a$*,* $x = b$ *equals*

$$P = -\int_a^b f(x)\,\mathrm{d}x\,.$$

Corollary 2 *Let* f *and* g *be continuous functions defined on an interval* $[a,b]$ *such that* $\forall x \in [a,b]:\; g(x) \le f(x)$*. Then the area* P *of the planar figure bounded by the graphs of* f *and* g *and by the vertical lines* $x = a$ *and* $x = b$ *equals*

$$P = \int_a^b (f(x) - g(x))\,\mathrm{d}x\,.$$

2.4.2 Lengths of curves

Consider a curve given by parametric equations

$$x = g(t),\; y = f(t), \quad t \in [a,b].$$

The length ℓ of such curve is computed as

$$\ell = \int_a^b \sqrt{(g'(t))^2 + (f'(t))^2}\,\mathrm{d}t.$$

2.4.3 Volumes of solids

Consider the planar figure bounded by the graph of a continuous nonnegative function f defined on an interval $[a,b]$ and by the lines $x = a$, $x = b$, $y = 0$. Let this figure rotate about the x-axis. The three-dimensional solid which we obtain in this way is called a *solid of revolution*. The volume V of such solid of revolution equals

$$V = \pi \int_a^b f^2(x)\,\mathrm{d}x.$$

2.4.4 Average value of a function

The *average* (or *mean*) *value* of a continuous function f defined on an interval $[a,b]$ is the number

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

The existence of the average value of continuous function is guaranteed by the following theorem:

Theorem 12 *If f is a continuous function on an interval $[a,b]$, then*

$$\exists\, c \in (a,b): \;\; f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

2.5 Numerical integration

Theorem 13 ***(Trapezoidal method)***
Let f be a continuous function on an interval $[a,b]$. Consider an equidistant division

$$a = x_0 < x_1 < \cdots < x_n = b$$

of $[a,b]$ with the step $h = \frac{b-a}{n}$ (the length of the intervals $[x_{i-1}, x_i]$, $i = 1,\ldots,n$) and denote $f(x_i) = y_i$ for $i = 0,1,\ldots n$. Then

$$\int_a^b f(x)\,\mathrm{d}x \doteq \frac{h}{2}\left(y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n\right).$$