1 Parametric equations of planar curves

Let $I \subseteq \mathbb{R}$ be an interval and let $\varphi_1, \varphi_2 : I \to \mathbb{R}$ be real functions. Then the map

$$\varphi : I \to \mathbb{R}^2$$

assigns to each $t \in I$ the pair $(\varphi_1(t), \varphi_2(t))$ of real numbers, i.e.

$$\varphi : t \mapsto (\varphi_1(t), \varphi_2(t)).$$

One refers to φ_1, φ_2 as the *coordinate functions of* φ.

We say that φ is *continuous on I* if the function φ_1 and φ_2 are continuous on I.

If the functions φ_1, φ_2 are differentiable on I, then one defines the *derivative* φ' of the map φ as

$$\varphi'(t) = (\varphi_1'(t), \varphi_2'(t)) \text{ for } t \in I.$$

Note that φ' is again a map from I to $\mathbb{R}^2$.

Definition 1 *Let $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ be a continuous map on I and let $\varphi'(t)$ exist for all $t \in I$ except for finitely many points $t_1, \ldots, t_k \in I$. Then the set $\mathcal{K}$ defined as the following set of points:*

$$\mathcal{K} = \{\varphi(t) | t \in I\} = \{(x,y) \in \mathbb{R}^2 | x = \varphi_1(t), y = \varphi_2(t), t \in I\}$$

refers to as a planar curve.
The map φ is called parametric representation (parametrization) *of the curve $\mathcal{K}$.*
The equations

$$\begin{aligned} x &= \varphi_1(t) \\ y &= \varphi_2(t), \quad t \in I \end{aligned}$$

are called parametric equations *of $\mathcal{K}$.*

Remark 1 *Parametric representation of a planar curve is not unique. There are infinitely many parameterizations of a given curve.*

Definition 2 *Let $\varphi : t \in I \mapsto (\varphi_1(t), \varphi_2(t)) \in \mathbb{R}^2$ be a parametrization of a curve $\mathcal{K}$. Then*

$$\vec{v}(t_0) := \varphi'(t_0) = (\varphi_1'(t_0), \varphi_2'(t_0))$$

is the tangent vector *to the curve $\mathcal{K}$ at $t = t_0$.*

- geometrical interpretation: the line passing through the point $\varphi(t_0)$ with the direction $\vec{v}(t_0)$ is the tangent line to the curve $\mathcal{K}$ at the point $\varphi(t_0)$
- physical interpretation: the vector $\vec{v}(t_0)$ is the vector of instantaneous velocity of an object currently positioned at the point $\varphi(t_0)$ but moving along the curve $\mathcal{K}$