1 Derivatives

Definition 1 *Let $x_0 \in \mathbb{R}$ and let f be a function defined on a neighbourhood $O(x_0)$. If there exists the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

f is said to be differentiable at the point x_0. *The value of the limit is called the* derivative of f at x_0 *and we denote it $f'(x_0)$. Thus,*

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

If this limit is proper/improper, then $f'(x_0)$ is said to be proper/improper derivative of f at x_0.

Note that f' is a function with the domain $D(f') = \{x \in D(f) | f'(x) \text{ exists and is proper}\}$.

Geometrical meaning of derivatives:
$f'(x_0)$ is a slope of the tangent line to the graph of a function f at the point x_0
$\rightsquigarrow$ the equation of the tangent line: $y - f(x_0) = f'(x_0)(x - x_0)$

Applications in physics, chemistry:
instantaneous velocity (of a moving body or reaction):

$$\frac{f(t_0 + \triangle t) - f(t_0)}{\triangle t} \xrightarrow[\triangle t \to 0]{} f'(t_0)$$

(average rate of change of function values) → (instantaneous rate of change of function values at time t_0)

Definition 2 *Let $x_0 \in \mathbb{R}$.*

- Right-hand derivative $f'_+(x_0)$ of a function f at x_0 *is defined as* $f'_+(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0}$ *if the limit exists.*
- Left-hand derivative $f'_-(x_0)$ of a function f at x_0 *is defined as* $f'_-(x_0) = \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0}$ *if the limit exists.*

Theorem 1 *A function f is differentiable at a point x_0 if and only if $f'_+(x_0) = f'_-(x_0)$. Then*

$$f'(x_0) = f'_+(x_0) = f'_-(x_0).$$

Definition 3 *(i) A function f has a* derivative on an open interval (a,b) (is differentiable on (a,b)) *if f is differentiable at each point of (a,b).*

(ii) A function f has a derivative on a closed interval $[a,b]$ *if f is differentiable on (a,b), differentiable from the right at a, and differentiable from the left at b.*

Theorem 2 *If a function f has proper derivative on (a,b), then f is continuous on (a,b).*

Note that continuous functions are not always differentiable (see e.g. $f(x) = |x|$ at $x_0 = 0$).

To compute derivatives of functions one needs to be able to differentiate elementary functions, see the table below, and to apply the rules for differentiation, see Theorem 3 and Definition 4.

Derivatives of elementary functions:

$(k)' = 0,\ k \in \mathbb{R}$	$(x^a)' = a \cdot x^{a-1},\ a \in \mathbb{R}$	$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$
$(a^x)' = a^x \cdot \ln(a),\ 1 \neq a > 0$	$(\log_a(x))' = \frac{1}{x \cdot \ln(a)},\ 1 \neq a > 0$	$(\arccos(x))' = \frac{-1}{\sqrt{1-x^2}}$
$(\sin(x))' = \cos(x)$	$(\cos(x))' = -\sin(x)$	$(\arctan(x))' = \frac{1}{1+x^2}$
$(\tan(x))' = \frac{1}{\cos^2(x)}$	$(\cot(x))' = \frac{-1}{\sin^2(x)}$	$(\text{arccot}(x))' = \frac{-1}{1+x^2}$

Theorem 3 *(i)* $[k \cdot f(x)]' = k \cdot f'(x)$, $k \in \mathbb{R}$

(ii) $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

(iii) $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

(iv) $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$, *for* $g(x) \neq 0$

(v) $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

Definition 4 *(derivatives of higher orders)*
Let $n \in \mathbb{N}$. *The* $n-$th derivative $f^{(n)}$ of a function f *is defined as* $f^{(n)} = \left[f^{(n-1)}\right]'$, *where* $f^{(0)} = f$.

Theorem 4

(i) Let a function f be right-hand side continuous at x_0 *and let* $\lim\limits_{x \to x_0+} f'(x)$ *exists. Then* $f'_+(x_0) = \lim\limits_{x \to x_0+} f'(x)$.

(ii) Let a function f be left-hand side continuous at x_0 *and let* $\lim\limits_{x \to x_0-} f'(x)$ *exists. Then* $f'_-(x_0) = \lim\limits_{x \to x_0-} f'(x)$.

1.1 Mean-value theorems

Theorem 5 ***Extreme value theorem***
Let f be a function continuous on $[a,b]$, $a,b \in \mathbb{R}$. *Then f attains a maximum and a minimum, both at least once, i.e.*

$$\exists c,d \in [a,b]\ \forall x \in [a,b]: \quad f(c) \leq f(x) \leq f(d).$$

Theorem 6 ***Rolle's mean-value theorem***
Let f be a function continuous on $[a,b]$ *and differentiable on* (a,b). *Further, let* $f(a) = f(b)$. *Then*

$$\exists c \in (a,b): \quad f'(c) = 0.$$

Theorem 7 ***Lagrange's mean-value theorem***
Let f be a function continuous on $[a,b]$ *and differentiable on* (a,b). *Then*

$$\exists c \in (a,b): \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 8 ***(Cauchy's mean-value theorem)***
Let f and g be functions continuous on $[a,b]$ *and differentiable on* (a,b). *Let* $g'(x) \neq 0$ *on* (a,b). *Then*

$$\exists c \in (a,b): \quad \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

1.2 L'Hospital's rule

Theorem 9 *Let*

(i) $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ *and* $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ *exists, or*

(ii) $\lim\limits_{x \to a} |g(x)| = +\infty$ *and* $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ *exists.*

Then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$.

Note that in the theorem above there is no restriction on a. Thus, $a \in \mathbb{R} \cup \{\pm\infty\}$. Further, the theorem holds for one-sided limits as well.

1.3 Graphing functions

1.3.1 Monotonicity

By applying the Lagrange's mean-value theorem one can derive the following characterization of monotonicity of a function.

Theorem 10 *Let f be a continuous and differentiable function on an interval I.*

(i) If $\forall x \in I: \ f'(x) > 0$, then f is increasing on I.

(ii) If $\forall x \in I: \ f'(x) \geq 0$, then f is non-decreasing on I.

(iii) If $\forall x \in I: \ f'(x) < 0$, then f is decreasing on I.

(iv) If $\forall x \in I: \ f'(x) \leq 0$, then f is non-increasing on I.

(v) If $\forall x \in I: \ f'(x) = 0$, then f is constant on I.

Theorem 11 *Let f be a continuous function on an interval I. Let there exist finitely many points $x_1, x_2, \ldots x_k \in I$ such that f is differentiable on $I \setminus \{x_1, \ldots, x_k\}$.*

(i) If $\forall x \in I \setminus \{x_1, \ldots, x_k\}: \ f'(x) > 0$, then f is increasing on I.

(ii) If $\forall x \in I \setminus \{x_1, \ldots, x_k\}: \ f'(x) \geq 0$, then f is non-decreasing on I.

(iii) If $\forall x \in I \setminus \{x_1, \ldots, x_k\}: \ f'(x) < 0$, then f is decreasing on I.

(iv) If $\forall x \in I \setminus \{x_1, \ldots, x_k\}: \ f'(x) \leq 0$, then f is non-increasing on I.

1.3.2 Local extrema

Definition 5 *Let f be a function defined on an interval (a,b). We say that f has a*

(i) local maximum *at $x_0 \in (a,b)$ if there exists $O(x_0)$ such that $\forall x \in O(x_0): \ f(x) \leq f(x_0)$,*

(ii) strict local maximum *at $x_0 \in (a,b)$ if there exists $\mathcal{P}(x_0)$ such that $\forall x \in \mathcal{P}(x_0): \ f(x) < f(x_0)$,*

(iii) local minimum *at $x_0 \in (a,b)$ if there exists $O(x_0)$ such that $\forall x \in O(x_0): \ f(x) \geq f(x_0)$,*

(iv) strict local minimum *at $x_0 \in (a,b)$ if there exists $\mathcal{P}(x_0)$ such that $\forall x \in \mathcal{P}(x_0): \ f(x) > f(x_0)$.*

Local maxima and minima of f are referred to as local extrema *of f.*

Theorem 12 *Let f be a continuous function on an interval (a,b) and let $x_0 \in (a,b)$.*

(i) If $\exists \mathcal{P}(x_0)$ such that $\forall x \in \mathcal{P}^-(x_0): \ f'(x) > 0$ and $\forall x \in \mathcal{P}^+(x_0): \ f'(x) < 0$, then f has a strict local maximum at x_0.

(ii) If $\exists \mathcal{P}(x_0)$ such that $\forall x \in \mathcal{P}^-(x_0): \ f'(x) < 0$ and $\forall x \in \mathcal{P}^+(x_0): \ f'(x) > 0$, then f has a strict local minimum at x_0.

(iii) If $f'(x_0) \neq 0$, then f does not have a local extremum at x_0.

Theorem 13 *Let f be a function defined on an interval (a,b) and let $x_0 \in (a,b)$.*

(i) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a strict local minimum at x_0.

(ii) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a strict local maximum at x_0.

1.3.3 Global extrema

Definition 6 *We say that a function f has a*

(i) global maximum *at* $x_0 \in D(f)$ *if* $\forall x \in D(f): f(x) \leq f(x_0)$.

(ii) global minimum *at* $x_0 \in D(f)$ *if* $\forall x \in D(f): f(x) \geq f(x_0)$.

Note that a function does not need to have a global maximum and/or minimum.

Theorem 14 *Let f be a continuous function defined on a closed interval $[a,b]$. Then f has a global maximum and a global minimum at a point of $[a,b]$.*

1.3.4 Convexity and concavity

Definition 7 *Let f be a continuous function on an interval I.*

(i) If for any $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ it holds that the point $P_2 = [x_2, f(x_2)]$ lies on or below the line joining the points $P_1 = [x_1, f(x_1)]$ and $P_3 = [x_3, f(x_3)]$, then we say that f is convex on *I.*

(ii) If for any $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ it holds that the point $P_2 = [x_2, f(x_2)]$ lies on or above the line joining the points $P_1 = [x_1, f(x_1)]$ and $P_3 = [x_3, f(x_3)]$, then we say that f is concave on *I.*

Theorem 15 *Let f be a function such that f'' is defined on an interval I (f has a second derivative on I).*

(i) If $f''(x) \geq 0$ on I, then f is convex on I.

(ii) If $f''(x) \leq 0$ on I, then f is concave on I.

Definition 8 *Let f be a function defined on (a,b) and let $x_0 \in (a,b)$. If*

- *f is continuous on (a,b),*
- *f has a derivative at x_0 (proper or improper),*
- *f is concave on (a,x_0) and convex on (x_0,b), or f is convex on (a,x_0) and concave on (x_0,b),*

then we say that f has an inflection point *at $(x_0, f(x_0))$.*

Theorem 16 *Let a function f have a second derivative f'' on an interval (a,b).*

(i) If there exists $\mathcal{P}(x_0)$ such that $\forall x \in \mathcal{P}^-(x_0): f''(x) > 0$ and $\forall x \in \mathcal{P}^+(x_0): f''(x) < 0$, or vice versa, then f has an inflection point at $(x_0, f(x_0))$.

(ii) If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then f has an inflection point at $(x_0, f(x_0))$.

Remark 1 *The points $x \in D(f)$ such that*

- *$f'(x) = 0$ are called* stationary points,
- *$f'(x) = 0$ or f' is not defined at x are called* critical points.

1.3.5 Asymptotes

Definition 9 *Let f be a function.*

- *If there exists $a \in \mathbb{R}$ such that $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$, then the line given by the equation $x = a$ is called a* vertical asymptote *of the graph of f.*
- *If $\lim_{x \to \infty} f(x) = b \in \mathbb{R}$ or $\lim_{x \to -\infty} f(x) = b \in \mathbb{R}$, then the line given by the equation $y = b$ is called a* horizontal asymptote *of the graph of f at ∞ or $-\infty$, respectively.*
- *If $\lim_{x \to \infty} (f(x) - kx - q) = 0$ or $\lim_{x \to -\infty} (f(x) - kx - q) = 0$ for some $k, q \in \mathbb{R}$, then the line given by the equation $y = kx + q$ is called an* oblique asymptote *of the graph of f at ∞ or $-\infty$, respectively.*

Theorem 17 *The line $y = kx + q$ is an oblique asymptote of the graph of a function f at $\pm\infty$ if and only if*

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = k \in \mathbb{R} \quad \text{and} \quad \lim_{x \to \pm\infty} (f(x) - kx) = q \in \mathbb{R}.$$

1.3.6 Graphing functions

To sketch a graph of a function f one follows the steps:

1. determine $D(f)$, decide on properties as being periodic, even, odd
2. discuss continuity of f, compute limits (one-sided limits) at the endpoints of the domain and at the points of discontinuity of f
3. specify the asymptotes of the graph of f
4. compute $f'(x)$ and based on it discuss monotonicity and local extrema of f
5. compute $f''(x)$ and based on it discuss convexity, concavity and inflection points of f
6. sketch the graph of f and determine $H(f)$

1.4 Newton's method - method of tangents

Newton's method is a method to approximate a root of a real-valued function f, i.e. to approximate a solution of the equation $f(x) = 0$. To ensure the existence of a root and the convergence of its approximations (speed and monotonicity is of interest) one assumes f has certain properties. We will formulate Newton's method for a continuous and twice differentiable function f. These assumptions imply quadratic and monotone convergence of the method.

Proposition 1 *Let f be a continuous function on an interval $[a,b]$ such that $f(a)f(b) < 0$. Then $\exists c \in (a,b)$ such that $f(c) = 0$ (such c is called a* root *of the equation $f(x) = 0$).*

Definition 10 *We say an interval $[a,b]$ is a* separation interval *of an equation $f(x) = 0$ if there exists a unique root α of this equation on $[a,b]$.*

Theorem 18 ***(Newton's method)***
Let f be a continuous function on an interval $[a,b]$ such that it has first and second derivative on $[a,b]$. Further, let us assume the interval $[a,b]$ is a separation interval of the equation $f(x) = 0$, i.e. we assume:

(i) $f(a)f(b) < 0$,

(ii) $\forall x \in [a,b] : f'(x) \neq 0$.

For the approximations x_n, constructed below, of the root $\alpha \in [a,b]$ of $f(x) = 0$ to converge monotonically to α we assume

(iii) $\forall x \in [a,b] : f''(x) \neq 0$.

Let us choose the 0-th approximation of α as

(iv) $x_0 \in [a,b]$ *such that* $f(x_0)\, f''(x_0) > 0$

and let us construct a sequence $\{x_n\}_{n=0}^{\infty}$ (x_n is referred to as n-th approximation of α) given by the following recursive formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Then

$$\lim_{n\to\infty} x_n = \alpha.$$

1.5 Taylor polynomial

Definition 11 *A function f of the form*

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where $a_i \in \mathbb{R}$ for $i = 0, 1, \ldots n$, $a_n \neq 0$, is called a polynomial of degree n *with constant coefficients. Note that $D(f) = \mathbb{R}$.*

Definition 12 *Let f be a function which has proper derivatives at a point x_0 up to order $n \geq 1$. Then the polynomial*

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the Taylor polynomial of degree n of f at x_0 *(or n-th order Taylor polynomial of f at x_0).*

Proposition 2 *For the Taylor polynomial $T_n(x)$ of a function f at a point x_0 it holds that*

$$T_n(x_0) = f(x_0), T_n'(x_0) = f'(x_0), \ldots, T_n^{(n)}(x_0) = f^{(n)}(x_0).$$

From the proposition above one can conclude that $f(x)$ and $T_n(x)$ have "similar behaviour" on a neighbourhood of x_0, i.e. $f(x) \doteq T_n(x)$ for $x \in O(x_0)$, which means that $T_n(x)$ approximates $f(x)$ close to x_0.

Theorem 19 ***(Taylor theorem)***
Let $f : \mathbb{R} \to \mathbb{R}$ be a function which has at $x_0 \in \mathbb{R}$ proper derivatives up to order $n \geq 1$. Then there exists a function $r : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = T_n(x) + r(x)(x - x_0)^n \quad \text{and} \quad \lim_{x \to x_0} r(x) = 0.$$

Taylor theorem describes the asymptotic behaviour of the remainder term

$$R_n(x) := f(x) - T_n(x) = r(x)(x - x_0)^n$$

which is the approximation error when approximating f with its Taylor polynomial. Under stronger regularity assumptions on f there are several precise formulae for the remainder term R_n of the Taylor polynomial, one of the most common ones is the Lagrange form of the remainder specified in the following theorem.

Theorem 20 *Let f be a $n+1$ times differentiable function on an interval I and let $T_n(x)$ be the Taylor polynomial of f at $x_0 \in I$. Then*

$$\forall x \in I \; \exists c \in (x, x_0) \; (\text{or } c \in (x_0, x)) : \; R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

i.e.

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

The last equality is called the Taylor formula.

1.6 Differential of a function

Definition 13

(i) The difference *of a function $f : \mathbb{R} \to \mathbb{R}$ of a single real variable x is the function Δf of two independent real variables x and Δx given by*

$$\Delta f(x, \Delta x) = f(x + \Delta x) - f(x).$$

(ii) The differential *of a function $f : \mathbb{R} \to \mathbb{R}$ of a single real variable x is the function $\mathrm{d}f$ of two independent real variables x and Δx given by*

$$\mathrm{d}f(x, \Delta x) = f'(x)\,\Delta x.$$

Remark 2

(i) *Instead of* $\mathrm{d}f(x,\Delta x)$ *one can write* $\mathrm{d}f(x)$.

(ii) *Since* $\mathrm{d}x(x,\Delta x) = 1\,\Delta x$, *it is conventional to write* $\Delta x = \mathrm{d}x$. *Hence* $\mathrm{d}f(x) = f'(x)\,\mathrm{d}x$.

(iii) *Note that from the Taylor formula it follows that* $\Delta f(x,\Delta x) = \mathrm{d}f(x,\Delta x) + R_1(x)$. *Thus,* $\mathrm{d}f(x,\Delta x)$ *approximates* $\Delta f(x,\Delta x)$ *and one can say that the differential is a linear approximation to the increment of a function.*