Handouts - week 2

1 Continuity and limits of functions

Terminology: Consider an arbitrary $a \in \mathbb{R}$ and an arbitrary $\varepsilon > 0$.

- The open interval $(a-\varepsilon, a+\varepsilon)$ is called the *ε-neighbourhood of the point a*. We denote it by $O_\varepsilon(a)$.
- The intervals $O_\varepsilon^+(a) = [a, a+\varepsilon)$ and $O_\varepsilon^-(a) = (a-\varepsilon, a]$ are referred to as a *right* and a *left ε-neighbourhood of a*, respectively.
- Neighbourhoods of the point a from which a itself is excluded are called *punctured* (sometimes *deleted*). Namely, $\mathcal{P}_\varepsilon(a) = O_\varepsilon(a) \setminus \{a\}$ denotes the *punctured ε-neighbourhood of a*, $\mathcal{P}_\varepsilon^+(a) = (a, a+\varepsilon)$ denotes the *punctured right ε-neighbourhood of a* and $\mathcal{P}_\varepsilon^-(a) = (a-\varepsilon, a)$ denotes the *punctured left ε-neighbourhood of a*.
- If x takes values arbitrarily close to a, we say *x approaches a* (or *x tends to a*) and we write $x \to a$. Similarly one defines the notation $x \to a+$, $x \to a-$, $x \to +\infty$, $x \to -\infty$.

1.1 Continuity of a function

Definition 1 *Let f be a real function of one real variable defined in a neighbourhood of a. We say that f is* continuous at $a \in D(f)$ *if*

$$\forall O_\varepsilon(f(a))\, \exists O_\delta(a): \; f(O_\delta(a)) \subseteq O_\varepsilon(f(a)).$$

Equivalently,

$$\forall \varepsilon > 0\, \exists \delta > 0: \; |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Definition 2 *We say that f is* continuous on an open interval (a,b) *if it is continuous at each point of (a,b).*

Definition 3 *We say that a function f is*

- continuous from the right (right-hand side continuous) at a point $a \in D(f)$ *if*

$$\forall O_\varepsilon(f(a))\, \exists O_\delta^+(a): \; f(O_\delta^+(a)) \subseteq O_\varepsilon(f(a)),$$

- continuous from the left (left-hand side continuous) at a point $a \in D(f)$ *if*

$$\forall O_\varepsilon(f(a))\, \exists O_\delta^-(a): \; f(O_\delta^-(a)) \subseteq O_\varepsilon(f(a)).$$

Definition 4 *We say that a function f is* continuous on a closed interval $[a,b]$ *if it is*

- *continuous at each point of (a,b),*
- *continuous from the right at the point a,*
- *continuous from the left at the point b.*

Theorem 1 *Let $a \in \mathbb{R}$ and let f and g be functions continuous at a. Then the functions $|f|$, $f \pm g$, $f \cdot g$ are continuous at a. Further, if $g(a) \neq 0$ then $\frac{f}{g}$ is continuous at a.*

Theorem 2 *If a function $y = f(x)$ is continuous at a point $x = a$ and a function $z = g(y)$ is continuous at the point $y = f(a)$, then the composition $(g \circ f)(x)$ is continuous at point $x = a$.*

1.2 Limit of a function

Definition 5 *Let $a \in \mathbb{R}$ and let $f : D(f) \to \mathbb{R}$ be a function defined in a punctured neighbourhood of a. We say that the* limit of the function $f(x)$ as x approaches a is equal to $A \in \mathbb{R}$ *(or that* the function f has the limit A at a*) and we write* $\lim_{x\to a} f(x) = A$ *if*

$$\forall O_\varepsilon(A)\ \exists \mathcal{P}_\delta(a) :\ \ f(\mathcal{P}_\delta(a)) \subset O_\varepsilon(A)$$

or equivalently

$$\forall \varepsilon > 0\ \exists \delta > 0 :\ \ 0 < |x-a| < \delta \Rightarrow |f(x) - A| < \varepsilon$$

Theorem 3 *A function has at a given point at most one limit.*

The following theorems are useful for calculating the limits:

Theorem 4 *Function f is continuous at a point $a \in D(f)$ if and only if $\lim\limits_{x\to a} f(x) = f(a)$.*

Theorem 5 *Let $f : D(f) \to \mathbb{R}$, $g : D(g) \to \mathbb{R}$, $a \in \mathbb{R}$. Then*

$$\exists \mathcal{P}(a) :\ (\forall x \in \mathcal{P}(a) :\ f(x) = g(x)) \Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

Theorem 6 *(so-called Squeeze or Sandwich theorem) Let the following conditions hold:*

- $\forall x \in \mathcal{P}(a) :\ g(x) \le f(x) \le h(x)$,
- $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x)$,

then there exists $\lim\limits_{x\to a} f(x)$ and it equals $\lim\limits_{x\to a} g(x)$.

Theorem 7 *Let $\lim\limits_{x\to a} f(x) = A \in \mathbb{R}$ and let $\lim\limits_{x\to a} g(x) = B \in \mathbb{R}$. Then*

(i) $\lim\limits_{x\to a}(f(x) \pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = A \pm B$,

(ii) $\lim\limits_{x\to a}(f(x)\cdot g(x)) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = A \cdot B$,

(iii) *if* $B \neq 0$ *then* $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = (\lim\limits_{x\to a} f(x))/(\lim\limits_{x\to a} g(x)) = \dfrac{A}{B}$.

Theorem 8 *Let $\lim\limits_{x\to a} g(x) = A \in \mathbb{R}$ and let f be a function continuous at A. Then*

$$\lim_{x\to a} f(g(x)) = f(A).$$

Remark 1 $\lim\limits_{x\to a}[f(x)]^{g(x)} = \lim\limits_{x\to a} \mathrm{e}^{g(x)\ln f(x)}$

1.3 One-sided limits of a function

Definition 6 *Let $a \in \mathbb{R}$ and let $f : D(f) \to \mathbb{R}$ be such that a punctured right neighbourhood of a is contained in $D(f)$. We say that f* has the right-sided limit $A \in \mathbb{R}$ at the point a *($\lim\limits_{x\to a+} f(x) = A$) if*

$$\forall O_\varepsilon(A)\ \exists \mathcal{P}_\delta^+(a) :\ \ f(\mathcal{P}_\delta^+(a)) \subset O_\varepsilon(A).$$

Definition 7 *Let $a \in \mathbb{R}$ and let $f : D(f) \to \mathbb{R}$ be such that a punctured left neighbourhood of a is contained in $D(f)$. We say that f* has the left-sided limit $A \in \mathbb{R}$ at the point a *($\lim\limits_{x\to a-} f(x) = A$) if*

$$\forall O_\varepsilon(A)\ \exists \mathcal{P}_\delta^-(a) :\ \ f(\mathcal{P}_\delta^-(a)) \subset O_\varepsilon(A).$$

Theorem 9 *$\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x)$. Then*

$$\lim_{x\to a} f(x) = \lim_{x\to a+} f(x) = \lim_{x\to a-} f(x).$$

The theorems stated above for two-sided limits hold for one-sided limits as well.

Theorem 10 *Let $f : D(f) \to \mathbb{R}$, $g : D(g) \to \mathbb{R}$, $a \in \mathbb{R}$.*

(i) f has at a at most one left-sided (right-sided) limit.

(ii) f is left-hand (right-hand) side continuous at a if and only if $\lim_{x\to a-} f(x) = f(a)$ ($\lim_{x\to a+} f(x) = f(a)$).

(iii) If $f = g$ on a punctured left (right) neighbourhood of a, then $\lim_{x\to a-} f(x) = \lim_{x\to a-} g(x)$ ($\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x)$).

(iv) Squeeze theorem:
If $\forall x \in \mathcal{P}^{\pm}(a) : g(x) \le f(x) \le h(x)$ and $\lim_{x\to a\pm} g(x) = \lim_{x\to a\pm} h(x)$, then $\lim_{x\to a\pm} f(x) = \lim_{x\to a\pm} g(x)$.

(v) $\lim_{x\to a\pm} (f(x)\pm g(x)) = \lim_{x\to a\pm} f(x) \pm \lim_{x\to a\pm} g(x)$

(vi) $\lim_{x\to a\pm} \dfrac{f(x)}{g(x)} = (\lim_{x\to a\pm} f(x))/(\lim_{x\to a\pm} g(x))$ if $\lim_{x\to a\pm} g(x) \neq 0$

(vii) If $\lim_{x\to a\pm} g(x) = A \in \mathbb{R}$ and f is left-hand (right-hand) side continuous at A, then $\lim_{x\to a\pm} f(g(x)) = f(A)$.

1.4 Limits involving infinity

Till now we have studied the limits $\lim_{x\to a} f(x) = L$, where $a, L \in \mathbb{R}$. Such limits are referred to as the *proper limits at proper points*. If $a = \pm\infty$, one considers the limits at plus/minus infinity. One can refer to them as *limits at improper points*. If $L = \pm\infty$, one says that the function $f(x)$ *diverges* at a. One calls such limits *improper*.

Definitions of improper limits of functions at proper/improper points are identical to the definition of proper limits of functions at proper points. However, one needs to recall that open (punctured) neighbourhoods of $-\infty$ are open intervals $(-\infty, a)$, $a \in \mathbb{R}\cup\{\infty\}$ and that open (punctured) neighbourhoods of ∞ are open intervals (a, ∞), $a \in \mathbb{R}\cup\{-\infty\}$. To make it clear we state the respective definitions below.

Definition 8 *Improper limits at proper points*
Let a function f be defined on a neighbourhood $\mathcal{P}(a)$ of $a \in \mathbb{R}$. Then

(i) $\lim_{x\to a} f(x) = \infty$ if $\forall K > 0\ \exists \mathcal{P}_\delta(a)\ \forall x \in \mathcal{P}_\delta(a) :\ f(x) > K$,

(ii) $\lim_{x\to a} f(x) = -\infty$ if $\forall L < 0\ \exists \mathcal{P}_\delta(a)\ \forall x \in \mathcal{P}_\delta(a) :\ f(x) < L$.

Remark 2 *By considering $\mathcal{P}^+_\delta(a)$ or $\mathcal{P}^-_\delta(a)$ instead of $\mathcal{P}_\delta(a)$ in Definition 8, one defines the respective one-sided improper limits at proper points.*

Definition 9 *Proper limits at improper points*

(i) Let $a \in \mathbb{R}\cup\{-\infty\}$ and let f be a function such that $(a,\infty) \subseteq D(f)$. We say that f has the proper limit $L \in \mathbb{R}$ at ∞ and write $\lim_{x\to\infty} f(x) = L$ if $\forall O_\varepsilon(L)\ \exists b > 0\ \forall x > b :\ f(x) \in O_\varepsilon(L)$.

(ii) Let $a \in \mathbb{R}\cup\{\infty\}$ and let f be a function such that $(-\infty, a) \subseteq D(f)$. We say that f has the proper limit L at $-\infty$ and write $\lim_{x\to-\infty} f(x) = L \in \mathbb{R}$ if $\forall O_\varepsilon(L)\ \exists b < 0\ \forall x < b :\ f(x) \in O_\varepsilon(L)$.

Definition 10 *Improper limits at improper points*

(i) Let $a \in \mathbb{R}\cup\{+\infty\}$ and let f be a function such that $(-\infty, a) \subseteq D(f)$. We say that f has the improper limit $+\infty$ at $-\infty$ and write $\lim_{x\to-\infty} f(x) = \infty$ if $\forall K > 0\ \exists b < 0\ \forall x < b :\ f(x) > K$.

(ii) *Let* $a \in \mathbb{R} \cup \{+\infty\}$ *and let* f *be a function such that* $(-\infty, a) \subseteq D(f)$. *We say that* f *has the improper limit* $-\infty$ *at* $-\infty$ *and write* $\lim_{x \to -\infty} f(x) = -\infty$ *if* $\forall L < 0\ \exists b < 0\ \forall x < b :\ f(x) < L$.

(iii) *Let* $a \in \mathbb{R} \cup \{-\infty\}$ *and let* f *be a function such that* $(a, \infty) \subseteq D(f)$. *We say that* f *has the improper limit* $+\infty$ *at* $+\infty$ *and write* $\lim_{x \to \infty} f(x) = \infty$ *if* $\forall K > 0\ \exists b > 0\ \forall x > b :\ f(x) > K$.

(iv) *Let* $a \in \mathbb{R} \cup \{-\infty\}$ *and let* f *be a function such that* $(a, \infty) \subseteq D(f)$. *We say that* f *has the improper limit* $-\infty$ *at* $+\infty$ *and write* $\lim_{x \to \infty} f(x) = -\infty$ *if* $\forall L < 0\ \exists b > 0\ \forall x > b :\ f(x) < L$.

The following theorems are useful for calculating (im)proper limits of functions at (im)proper points.

Theorem 11 *Let a function* f *be defined on a neighbourhood* $\mathcal{P}(a)$ *of* $a \in \mathbb{R}$. *Then*

(i) $\lim_{x \to a} f(x) = \infty \quad \Leftrightarrow \quad \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = \infty$

(ii) $\lim_{x \to a} f(x) = -\infty \quad \Leftrightarrow \quad \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = -\infty$

Theorem 12 *Let* f *and* g *be functions such that* $f = g$ *on a neighbourhood* $\mathcal{P}(a)$ *of* $a \in \mathbb{R} \cup \{+\infty, -\infty\}$. *Then* $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

Theorem 13 *Let* $\lim_{x \to \pm\infty} f(x) = A \in \mathbb{R}$ *and* $\lim_{x \to \pm\infty} g(x) = B \in \mathbb{R}$. *Then*

(i) $\lim_{x \to \pm\infty} (f(x) \pm g(x)) = \lim_{x \to \pm\infty} f(x) \pm \lim_{x \to \pm\infty} g(x) = A \pm B$,

(ii) $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = \lim_{x \to \pm\infty} f(x) \cdot \lim_{x \to \pm\infty} g(x) = A \cdot B$,

(iii) *if* $B \neq 0$, *then* $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to \pm\infty} f(x)}{\lim_{x \to \pm\infty} g(x)} = \dfrac{A}{B}$.

Theorem 14 *Consider functions* f *and* g *defined on a neighbourhood* $\mathcal{P}(a)$ *of* $a \in \mathbb{R} \cup \{+\infty, -\infty\}$.

(i) *If* $\lim_{x \to a} f(x) = \infty$ *and* $\lim_{x \to a} g(x) = \infty$, *then* $\lim_{x \to a} (f(x) + g(x)) = \infty$ *and* $\lim_{x \to a} (f(x) \cdot g(x)) = \infty$.

(ii) *If* $\lim_{x \to a} f(x) = -\infty$ *and* $\lim_{x \to a} g(x) = -\infty$, *then* $\lim_{x \to a} (f(x) + g(x)) = -\infty$ *and* $\lim_{x \to a} (f(x) \cdot g(x)) = \infty$.

(iii) *If* $\lim_{x \to a} f(x) = \infty$ *and* $\lim_{x \to a} g(x) = -\infty$, *then* $\lim_{x \to a} (f(x) \cdot g(x)) = -\infty$.

(iv) *If* $\lim_{x \to a} f(x) = A \in \mathbb{R}, A > 0$ *and* $\lim_{x \to a} g(x) = \pm\infty$, *then* $\lim_{x \to a} (f(x) \cdot g(x)) = \pm\infty$.

(v) *If* $\lim_{x \to a} f(x) = A \in \mathbb{R}$ *and* $\lim_{x \to a} g(x) = \pm\infty$, *then* $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$.

The rules to calculate limits according to Theorem 14 can be easily remembered in the following form:

$\infty + \infty = \infty$	$-\infty \cdot \infty = -\infty$
$\infty \cdot \infty = \infty$	$A \cdot \infty = \infty$ when $A > 0$
$-\infty + (-\infty) = -\infty$	$A \cdot (-\infty) = -\infty$ when $A > 0$
$-\infty \cdot (-\infty) = \infty$	$\frac{A}{\pm\infty} = 0$ when $A \in \mathbb{R}$

Theorem 15 *Let* f *be a function bounded on a neighbourhood* $P(a)$ *of* $a \in \mathbb{R} \cup \{+\infty, -\infty\}$. *Then the following holds:*

(i) *if* $\lim_{x \to a} g(x) = \pm\infty$, *then* $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$,

(ii) *if* $\lim_{x \to a} g(x) = 0$, *then* $\lim_{x \to a} f(x)g(x) = 0$.

Theorem 16 *Consider functions* f *and* g *defined on a neighbourhood* $\mathcal{P}(a)$ *of* $a \in \mathbb{R} \cup \{+\infty, -\infty\}$. *Let* $\lim_{x \to a} f(x) = A > 0$ *and let* $\lim_{x \to a} g(x) = 0$.

(i) *If* $g(x) > 0$ *on* $\mathcal{P}(a)$*, then* $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = +\infty$.

(ii) *If* $g(x) < 0$ *on* $\mathcal{P}(a)$*, then* $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = -\infty$.

(iii) *If the function* g *takes positive and negative values on every neighbourhood* $P(a)$*, then* $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ *does not exist.*

Mnemonics for Theorem 16:

$\frac{A}{0_+} = +\infty$ when $A > 0$	$\frac{A}{0_-} = -\infty$ when $A > 0$

Theorem 17 *Squeeze or Sandwich theorem:*
Let $a \in \mathbb{R} \cup \{+\infty, -\infty\}$ *and let the functions* f, g, h *be defined on a neighbourhood* $\mathcal{P}(a)$.

- *If* $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = L \in \mathbb{R}$ *and* $\forall x \in \mathcal{P}(a):\ g(x) \le f(x) \le h(x)$*, then* $\lim\limits_{x\to a} f(x) = L$.
- *If* $\lim\limits_{x\to a} g(x) = \infty$ *and* $\forall x \in \mathcal{P}(a):\ g(x) \le f(x)$*, then* $\lim\limits_{x\to a} f(x) = \infty$.
- *If* $\lim\limits_{x\to a} h(x) = -\infty$ *and* $\forall x \in \mathcal{P}(a):\ f(x) \le h(x)$*, then* $\lim\limits_{x\to a} f(x) = -\infty$.

Note that the following expressions are indefinite, they depend on instances.

$\infty - \infty$	$0 \cdot \infty$	$\frac{\infty}{\infty}$	$\frac{0}{0}$	1^∞	0^0	∞^0

1.5 Limits of sequencies

Definition 11 *We define a* sequence $\{a_n\}_{n=1}^\infty$ *to be a function whose domain is a subset of* $\mathbb{N}$ *(or more generally of* $\mathbb{Z}$*) and whose codomain is* $\mathbb{R}$*. Namely,* $\{a_n\}_{n=1}^\infty : n \in \mathbb{N} \mapsto a_n \in \mathbb{R}$*. The values* a_n *are called the* elements *(or terms, members) of the sequence* $\{a_n\}_{n=1}^\infty$*,* n *is called the* index *of the element* a_n.

Examples:
Arithmetic sequence: $a_n = a_1 + (n-1)d$, where $n \in \mathbb{N}$ and $d \in \mathbb{R}$ is a common difference
Geometric sequence: $a_n = a_1 \cdot q^{n-1}$, where $n \in \mathbb{N}$ and $q \in \mathbb{R} \setminus \{0\}$ is a common ratio (or quotient)

Definition 12 *We say that a sequence* $\{a_n\}_{n=1}^\infty$ *has a limit* A*, i.e.* $\lim\limits_{n\to\infty} a_n = A$*, if*

- *in case* $A \in \mathbb{R}$*:* $\forall O_\varepsilon(A)\ \exists n_0 \in \mathbb{N}\ \forall n \ge n_0:\ a_n \in O_\varepsilon(A)$,
- *in case* $A = +\infty$*:* $\forall K > 0\ \exists n_0 \in \mathbb{N}\ \forall n \ge n_0:\ a_n > K$,
- *in case* $A = -\infty$*:* $\forall L < 0\ \exists n_0 \in \mathbb{N}\ \forall n \ge n_0:\ a_n < L$.

A sequence $\{a_n\}_{n=1}^\infty$ *is called* convergent *if it has a proper limit, i.e.* $\lim\limits_{n\to\infty} a_n \in \mathbb{R}$.
A sequence $\{a_n\}_{n=1}^\infty$ *is called* divergent *if* $\lim\limits_{n\to\infty} a_n = \pm\infty$ *or if* $\lim\limits_{n\to\infty} a_n$ *does not exist.*

Definition 13 *Consider a sequence* $\{a_n\}_{n=1}^\infty$.

(i) *If* $\forall n \in \mathbb{N}:\ a_n < a_{n+1}$*, we say that* $\{a_n\}_{n=1}^\infty$ *is* increasing.

(ii) *If* $\forall n \in \mathbb{N}:\ a_n \le a_{n+1}$*, we say that* $\{a_n\}_{n=1}^\infty$ *is* non-decreasing.

(iii) *If* $\forall n \in \mathbb{N}:\ a_n > a_{n+1}$*, we say that* $\{a_n\}_{n=1}^\infty$ *is* decreasing.

(iv) *If* $\forall n \in \mathbb{N}:\ a_n \ge a_{n+1}$*, we say that* $\{a_n\}_{n=1}^\infty$ *is* non-increasing.

Theorem 18 *A decreasing or non-increasing sequence is bounded above. An increasing or non-decreasing sequence is bounded below.*

Theorem 19 *A monotone sequence has always limit. If the sequence is bounded, then the limit is proper.*

Remark 3 *One can prove that there exists a limit of the sequence* $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$*. This limit is used to define* Euler's number e.