4 Real functions of one real variable

Intuitively: A *function* is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

Definition 4 *A function f from X to Y is a subset of the Cartesian product $X \times Y$ subject to the following condition: for every x in X there is exactly one element y in Y such that the ordered pair (x, y) is contained in the subset defining the function f.*

We use the notation: $y = f(x)$, $f : X \to Y$, $x \mapsto f(x)$, $f : x \mapsto y$. In the expression $f(x)$, x is the argument *and $f(x)$ is the* value *of the function f.*

The set X is called the domain of f, *we denote it by $D(f)$. The* natural domain *of a function is the maximal set of values for which the function is defined.*

The set Y is called the codomain of f.

The image *(sometimes called the* range*) of f, denoted by $H(f)$, is the set of all values assumed by f for all possible x, i.e. $H(f) = \{f(x)|x \in D(f)\}$.*

A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is a *real function of one real variable.*

Functions can be specified by formulae, algorithms, tables, graphs, ...

Definition 5 *The* graph *of a function f is the set of all ordered pairs $(x, f(x))$, where $x \in D(f)$.*

Remark 1 *Two functions f and g equal ($f = g$) if*

(i) $D(f) = D(g)$

(ii) $\forall x \in D(f) : f(x) = g(x)$

5 Operations on functions

Definition 6 *Let f, g be real functions of one real variable.*

- *The* sum *of f and g is the function $h = f + g$ such that*
  (i) $D(h) = D(f) \cap D(g)$,
  (ii) $\forall x \in D(h) : h(x) = f(x) + g(x)$.
- *The* product *of f and g is the function $h = f.g$ such that*
  (i) $D(h) = D(f) \cap D(g)$,
  (ii) $\forall x \in D(h) : h(x) = f(x).g(x)$.
- *The* quotient *of f and g is the function $h = \frac{f}{g}$ such that*
  (i) $D(h) = (D(f) \cap D(g)) \setminus N(g)$, *where* $N(g) = \{x \in D(g)|g(x) = 0\}$,
  (ii) $\forall x \in D(h) : h(x) = \frac{f(x)}{g(x)}$.
- *Let $f : X \to Y$ and $g : Y \to Z$. Then the function $h = g \circ f : X \to Z$ defined as*
  (i) $D(h) = \{x \in D(f)|f(x) \in D(g)\}$,
  (ii) $\forall x \in D(h) : h(x) = g(f(x))$,

  is called the composition *of f with g. We refer to g as the* outer *function and to f as the* inner *function.*

Remark 2 *It is not in general true that $g \circ f = f \circ g$.*

6 Basic properties of functions

Definition 7 *A function* $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ *is* injective *on* $M \subseteq D(f)$ *iff*

$$\forall x_1, x_2 \in M : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

- Equivalent formulation (used to prove that f is injective):

$$\forall x_1, x_2 \in M : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

- Negation (used to prove that f is not injective):

$$\exists x_1, x_2 \in M : x_1 \neq x_2 \wedge f(x_1) = f(x_2)$$

Definition 8 *A function* $f : X \to Y$ *is* surjective *iff* $\forall y \in Y \; \exists x \in X : \; y = f(x)$.

Definition 9 *Consider a function* $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ *and a set* $M \subseteq D(f)$. *If for all* $x_1, x_2 \in M$, $x_1 < x_2$ *it holds that*

(i) $f(x_1) < f(x_2)$, *then* f *is* increasing *on* M,

(ii) $f(x_1) \leq f(x_2)$, *then* f *is* non-decreasing *on* M,

(iii) $f(x_1) > f(x_2)$, *then* f *is* decreasing *on* M,

(iv) $f(x_1) \geq f(x_2)$, *then* f *is* non-increasing *on* M.

If f *satisfies any of* $(i) - (iv)$, *it is called* monotone. *If* f *satisfies either* (i) *or* (iii), *it is called* strictly monotone.

Proposition 1 *The sum of two increasing (decreasing) functions is an increasing (decreasing) function.*

Theorem 1 *Let* f *be a strictly monotone function on a set* $M \subseteq \mathbb{R}$. *Then* f *is injective on* M.

Definition 10 *We say that a function* f *is*

(i) bounded below *on* $D(f)$ *iff* $\exists b \in \mathbb{R} \; \forall x \in D(f) : \; b \leq f(x)$,

(ii) bounded above *on* $D(f)$ *iff* $\exists a \in \mathbb{R} \; \forall x \in D(f) : \; f(x) \leq a$,

(iii) bounded *on* $D(f)$ *iff it is bounded below and above on* $D(f)$.

Definition 11 *We say that a function* f *is*

(i) even *iff* $x \in D(f) \Leftrightarrow -x \in D(f)$ *and* $\forall x \in D(f) : \; f(x) = f(-x)$,

(ii) odd *iff* $x \in D(f) \Leftrightarrow -x \in D(f)$ *and* $\forall x \in D(f) : \; f(x) = -f(-x)$.

Remark 3

(i) The graph of an even function is symmetric w.r.t. y-axis.

(ii) The graph of an odd function is symmetric w.r.t. the origin.

(iii) Domains of even and odd functions are symmetric w.r.t. the origin.

Definition 12 *A function* $f : D(f) \to \mathbb{R}$ *is said to be* periodic *iff* $\exists p \in \mathbb{R}$, $p \neq 0$ *such that*

(i) $x \in D(f) \Rightarrow x \pm p \in D(f)$,

(ii) $\forall x \in D(f) : \; f(x \pm p) = f(x)$.

p *is a* period *of* f. *The smallest positive period is called the* fundamental period *(or primitive period, basic period, prime period).*

Theorem 2

(i) Let f *be a periodic function with period* p *and let* g *be a function such that* $H(f) \subseteq D(g)$. *Then the function* $g \circ f$ *is periodic with period* p.

(ii) Let f *be a periodic function with period* p *and let* $a \in \mathbb{R}$, $a \neq 0$. *Then the function* $g(x) = f(ax)$ *is periodic with period* $\frac{p}{a}$.

6.1 Inverse functions

Definition 13 *Let f be an injective function. The* inverse function *of f is the function f^{-1} defined as follows*

- $D(f^{-1}) = H(f)$
- $H(f^{-1}) = D(f)$
- $y = f(x) \Leftrightarrow x = f^{-1}(y)$

Remark 4

(i) *The graph of f^{-1} can be obtained from the graph of f by switching the positions of the x and y axes. This is equivalent to reflecting the graph across the line $y = x$.*

(ii) $\forall x \in D(f) : f^{-1}(f(x)) = x$

(iii) $\forall y \in D(f^{-1}) = H(f) : f(f^{-1}(y)) = y$

(iv) $(f^{-1})^{-1} = f$

Remark 5 *Exponential and logarithmic functions*

- $y = a^x \Leftrightarrow x = \log_a(y), x \in \mathbb{R}, y > 0, 1 \neq a > 0$
- *Note that* $f(x)^{g(x)} = e^{g(x)\ln(f(x))}$.

Theorem 3 *Properties of inverse trigonometric functions:*

$f(x)$	$\arcsin(x)$	$\arccos(x)$	$\arctan(x)$	$\operatorname{arccot}(x)$
$D(f)$	$\langle -1, 1\rangle$	$\langle -1, 1\rangle$	$\mathbb{R}$	$\mathbb{R}$
$H(f)$	$\langle -\frac{\pi}{2}, \frac{\pi}{2}\rangle$	$\langle 0, \pi\rangle$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(0, \pi)$
increasing	✓	–	✓	–
decreasing	–	✓	–	✓
even	–	–	–	–
odd	✓	–	✓	–
$f^{-1}(x)$	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\cot(x)$
	$x \in \langle -\frac{\pi}{2}, \frac{\pi}{2}\rangle$	$x \in \langle 0, \pi\rangle$	$x \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$x \in (0, \pi)$

Theorem 4

(i) $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ *for* $x \in \langle -1, 1\rangle$

(ii) $\arctan(x) + \operatorname{arccot}(x) = \frac{\pi}{2}$ *for* $x \in \mathbb{R}$