2. Numbers

2.1 Number sets

Numbers are mathematical objects used to count, measure and label. There are several number systems which were discovered throughout history. The most basic system is the set of natural numbers which are followed by integers, rational numbers, irrational numbers, real numbers and complex numbers.

2.1.1 Natural numbers

Natural numbers $1,2,3,4,\ldots$ are denoted by $\mathbb{N}$. It is sometimes useful to include zero in the set of natural numbers and in this case the notation $\mathbb{N}_0$ is used. Hence,

$$\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0,1,2,\ldots\}.$$

There are infinitely many natural numbers. Natural numbers are closed under addition and multiplication, i.e. if we add or multiply natural numbers, we will get a natural number as a result.

2.1.2 Integers

One refers to the set which contains natural numbers, their negative counterparts and zero as the set of *integers*. The set of integers is denoted by $\mathbb{Z}$ (from German *die Zahlen* meaning the numbers), i.e.

$$\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}.$$

Sometimes natural numbers are called *positive integers*, natural numbers with zero are called *non-negative integers* and the set $\mathbb{Z} \setminus \mathbb{N}_0$ is referred to as *negative integers*. Note that the negative of a positive integer is a number that produces 0 when it is added to the corresponding positive integer, e.g. -1 is the negative of 1.

There are infinitely many integers, they are closed under addition, multiplication and subtraction.

2.1.3 Rational numbers

A *rational number* is a number that can be expressed as a fraction with an integer numerator and a natural number denominator. The set of all rational numbers is denoted by $\mathbb{Q}$ (from quotient) and it is defined by

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{Z}, q \in \mathbb{N} \right\}.$$

Note that various fractions may represent the same rational number, e.g. $1/3, 2/6, 3/9$ are all equal. In particular, $\frac{p}{q} = \frac{r}{s}$ if and only if $p\,s = r\,q$.

There are infinitely many rational numbers. To be more precise, we say that the set of rational numbers is countably infinite which means that there are as many rational numbers as there are natural numbers. In fact, the number of elements (or, more precisely, the cardinality) of the sets $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ is the same. However, the next proposition demonstrates that this notion of infinity is not as simple as one might initially think.

Proposition 2.1.1 There are infinitely many rational numbers between two arbitrary rational numbers.

Proof. Let $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ be two arbitrary rational numbers such that $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. For any $n \in \mathbb{N}$ define a_n by

$$a_n := \frac{p_1}{q_1} + \frac{\frac{p_2}{q_2} - \frac{p_1}{q_1}}{2^n} = \frac{p_1}{q_1} + \frac{p_2 q_1 - p_1 q_2}{2^n q_1 q_2} = \frac{p_1 q_2 (2^n - 1) + p_2 q_1}{2^n q_1 q_2}.$$

The numerator $p_1 q_2 (2^n - 1) + p_2 q_1$ is an integer since p_1, p_2, q_1, q_2 and $2^n - 1$ are integers and integers are closed under multiplication and addition. Similarly, the denominator $2^n q_1 q_2$ is a natural number. It follows that $a_n \in \mathbb{Q}$ and, moreover, we have that a_n lies between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$. Since n is used as the index of a_n and there are infinitely many of n's in $\mathbb{N}$, there are infinitely many a_n's such that

$$\frac{p_1}{q_1} < a_n < \frac{p_2}{q_2}.$$

■

Rational numbers are closed under addition, multiplication, subtraction and division by non-zero numbers.

2.1.4 Real numbers

Real numbers are used to measure continuous quantities; and they can be expressed by decimal representations that have an infinite sequence of digits to the right of the decimal point. Each consecutive digit in the decimal representation is measured in units one tenth the size of the previous one, e.g.

$$73.621 = 7 \cdot 10^1 + 3 \cdot 10^0 + 6 \cdot 10^{-1} + 2 \cdot 10^{-2} + 1 \cdot 10^{-3}.$$

The set of real numbers is denoted by $\mathbb{R}$. In the same way as rational numbers, real numbers are closed under addition, multiplication, subtraction and division by non-zero numbers.

One can represent the set $\mathbb{R}$ geometrically as points on an infinitely long line called the real line, see Figure 2.1.

From the description above one may get the wrong impression that there is not much of a difference between rational and real numbers. However, the difference is enormous as there are a

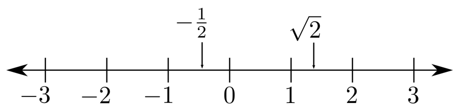

Figure 2.1: Real line

lot more real numbers than rational numbers. More precisely, both the sets $\mathbb{Q}$ and $\mathbb{R}$ are infinite, but the reals are uncountable. In fact, the cardinality of $\mathbb{Q}$ equals the cardinality of $\mathbb{N}$, but the cardinality of $\mathbb{R}$ equals that of the set of subsets of $\mathbb{N}$, and the so-called Cantor's diagonal argument states that the cardinality of the latter set is strictly greater than the cardinality of $\mathbb{N}$.

■ **Example 2.1** To demonstrate that not all real numbers are rational, let us show that the number $\sqrt{2}$ is not rational.

Let us assume for a contradiction that $\sqrt{2}$ is rational. That means that it can be written as a fraction of $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, i.e.

$$\sqrt{2} = \frac{p}{q}.$$

Without loss of generality, we may assume that p and q have no common factors. By taking the square of both sides of the equation above, we obtain

$$2 = \frac{p^2}{q^2} \quad \text{and thus} \quad p^2 = 2q^2.$$

The last equality means that p^2 is even. The only way this can be true is that p itself is even which means that p^2 has to be divisible by 4. Hence, q^2 and therefore q, must be even. So, p and q are both even which is a contradiction to the assumption that they do not have common factors. ■

Another differences between $\mathbb{R}$ and $\mathbb{Q}$ follow from more rigorous definitions of real numbers. In particular, real numbers can be constructed by *completion* of rational numbers. Completing rational numbers can be done in several ways, e.g. by adding new elements to $\mathbb{Q}$ so that

- the least upper bounds of all non-empty subsets in $\mathbb{Q}$ are included, see Section 2.2 for the notion of an upper bound, or so that
- the limits of all rational sequences are included, see Section 4.0.1 for the notion of the limit of a sequence.

■ **Example 2.2** The set $\mathbb{Q}$ does not have the least-upper-bound property. For $\mathbb{Q}$ to have this property it is not enough for some sets in $\mathbb{Q}$ to have their least upper bounds in $\mathbb{Q}$ (e.g. the least upper bound for the set $\{q \in \mathbb{Q} | 0 < q < 1\}$ is 1 which is a rational number), but this has to be true for every set in $\mathbb{Q}$. A good counterexample is given by the set $\{q \in \mathbb{Q} \,|\, q^2 < 2\}$ which has the least upper bound $\sqrt{2}$ but $\sqrt{2} \notin \mathbb{Q}$. ■

■ **Example 2.3** Let us construct a sequence of rational numbers which converges to a number which is not rational. In particular, let us find a rational sequence converging to $\sqrt{2}$. Note that the decimal expansion of $\sqrt{2}$ looks like

$$\sqrt{2} = 1.41421356237310\ldots.$$

A sequence that converges to $\sqrt{2}$ then can be chosen as

$$1, 1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

Basically, one just adds a digit each time. This is clearly a sequence of rational numbers because every decimal number with a finite decimal expansion is rational, and it clearly converges to $\sqrt{2}$.

Note that one could formulate the same sequence as $\{a_n\}_{n=0}^{\infty}$ where

$$a_n := 10^{-n} \max\{k \in \mathbb{Z} \,|\, k^2 \leq 2 \cdot 10^{2n}\}.$$

■

2.1.5 Irrational numbers

Irrational numbers, denoted by $\mathbb{I}$, are all real numbers which are not rational numbers, i.e.

$$\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}.$$

As an example of irrational numbers, one can list $\sqrt{2}, e, \pi$. Note that square root of any natural number except the ones which are the so-called perfect squares (e.g. $1, 4, 9, 16, \ldots$) is an irrational number.

Since irrational numbers are special case of real numbers, they can be expressed by decimal representations that have an infinite sequence of digits to the right of the decimal point. However, their decimal representations are special in the sense that they do not contain a subsequence of digits whose repetition makes up the tail of the representation. For example, the decimal representation of the number π starts with 3.14159265358979 but no finite number of digits can represent π exactly. Note that the decimal representation of a rational number is either finite (the number of digits after the decimal point is finite) or infinite (it contains a subsequence of digits whose repetition makes up the tail of the representation).

Irrational numbers are not closed under addition and subtraction; their sums and differences could be both rational (e.g. $\pi - \pi = 0 \in \mathbb{Q}$) and irrational (e.g. $\pi + \pi = 2\pi \in \mathbb{I}$). Moreover, it is not always possible to decide whether the answer is a rational or an irrational number as in the case of $e + \pi$ and $e - \pi$. Furthermore, irrational numbers are not closed under multiplication and division by non-zero numbers. The products and quotients could be rational (e.g. $e/e = 1 \in \mathbb{Q}$), irrational (e.g. $\sqrt{2}\sqrt{3} = \sqrt{6} \in \mathbb{I}$) or of a type which is not known yet (e.g. it is still an open problem to decide whether $e\pi$ is a rational or an irrational number).

Recall that there are uncountably many real numbers and countably many rational numbers. Therefore, there are uncountably many irrational numbers which makes almost all real numbers irrational.

2.1.6 Complex numbers

Real numbers are not algebraically closed, that is, not all polynomials with real coefficients have real roots. For example, to find the roots of the polynomial $x^2 + 1$ we have to solve the equation $x^2 + 1 = 0$. Then, $x = \pm\sqrt{-1}$, but $\sqrt{-1} \notin \mathbb{R}$. Thus, this equation has no solution in $\mathbb{R}$ which means that the polynomial $x^2 + 1$ has no real roots. By introducing a new number system containing $\sqrt{-1}$, the so-called *complex numbers*, one obtains a number system which is algebraically closed (all polynomials with complex coefficients have complex roots). In fact, the idea to "extend" the real numbers with a symbol representing $\sqrt{-1}$ first appeared when mathematicians tried to find real solutions to cubic equations with real coefficients.

A *complex number* is a number of the form $a + b \cdot i$ where a and b are real numbers and i is an indeterminate satisfying $i^2 = -1$. The indeterminate i is called the *imaginary unit*. The set of all complex numbers is denoted by $\mathbb{C}$,

$$\mathbb{C} = \{z = a + bi | a, b \in \mathbb{R}\}.$$

Hence, complex numbers may be defined as binomials in the single indeterminate i with the relation $i^2 + 1 = 0$ imposed. By this definition, complex numbers can be added, subtracted or multiplied using the rules for addition and multiplication of polynomials. In particular, for $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i \in \mathbb{C}$, the following holds:

$$z_1 \pm z_2 = (a_1 \pm a_2) + (b_1 \pm b_2)i,$$

$$z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i.$$

■ **Example 2.4** Consider two complex numbers: $2+3i$ and $3-4i$. Then

$$(2+3i)+(3-4i) = (2+3)+(3-4)i = 5-i$$

and

$$(2+3i)\cdot(3-4i) = 2\cdot(3-4i)+3i\cdot(3-4i) = 6-8i+9i-12i^2 = 6+12+(-8+9)i = 18+i.$$

■

Furthermore, complex numbers can also be divided by nonzero complex numbers. The division is based on the following computation:

$$\frac{z_1}{z_2} = \frac{a_1+b_1 i}{a_2+b_2 i}\cdot\frac{a_2-b_2 i}{a_2-b_2 i} = \frac{a_1 a_2 + b_1 b_2 + (b_1 a_2 - a_1 b_2)i}{a_2^2+b_2^2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2+b_2^2} + \frac{b_1 a_2 - a_1 b_2}{a_2^2+b_2^2} i.$$

For z_2 to be nonzero, a_2, b_2, or both a_2 and b_2 have to be nonzero. Let us point out that the "trick" to divide two complex numbers $a_1+b_1 i$ and $a_2+b_2 i$ by multiplying their fraction $\frac{a_1+b_1 i}{a_2+b_2 i}$ by one in the specific form $\frac{a_2-b_2 i}{a_2-b_2 i}$ is based on the fact that the product $(a_2+b_2 i)(a_2-b_2 i)$ is the real number $a_2^2+b_2^2$.

■ **Example 2.5** To divide $2+3i$ by $3-4i$ one proceeds as follows:

$$\frac{2+3i}{3-4i} = \frac{2+3i}{3-4i}\cdot\frac{3+4i}{3+4i} = \frac{6+8i+9i-12}{9+16} = -\frac{6}{25}+\frac{17}{25}i.$$

■

The real number a is called the *real part* of the complex number $z = a+bi$ and it is denoted by $\mathrm{Re}(z)$. The real number b is called the *imaginary part* of $z = a+bi \in \mathbb{C}$ and it is denoted by $\mathrm{Im}(z)$.

■ **Example 2.6** We have that

$$\mathrm{Re}(3-4i) = 3 \quad \text{and} \quad \mathrm{Im}(3-4i) = -4.$$

■

Note that a real number a can be regarded as a complex number $a+0i$ whose imaginary part is 0. A *purely imaginary* number bi is a complex number $0+bi$ whose real part is zero. It is common to write a for $a+0i$ and bi for $0+bi$.

We can visualize complex numbers pictorially in a similar way as we did real numbers. One-dimensional real line is a geometric representation of real numbers. Geometrically, complex numbers extend this concept to the two-dimensional complex plane by using the horizontal axis for the real part and the vertical axis for the imaginary part. The complex number $z = a+bi$ can be identified with the point $(a = \mathrm{Re}(z), b = \mathrm{Im}(z))$ in the complex plane, see Figure 2.2. Such representation of z is referred to as the *Cartesian form* of z. Hence, one can write

$$\mathbb{C} = \{z = (a,b) \,|\, a,b \in \mathbb{R}\}$$

with the respective rules for addition, multiplication and division. We can translate the Cartesian form (a,b) of a complex number $z = a+bi$ into polar coordinates. Then, the position (a,b) in the complex plane is given by the *magnitude* $r = \sqrt{a^2+b^2}$ of z, i.e. the distance of the point (a,b) from the origin $(0,0)$ which also defines the absolute value $|z| = |a+bi| := \sqrt{a^2+b^2}$ of z, and by the angle θ called the *argument* of z and denoted by arg z which is the angle subtended between the positive real axis and the line segment bounded by the end points $(0,0)$ and (a,b) in a counterclockwise direction, see Figure 2.2. The relation between (a,b) and (r,θ) is given by

$$a = r\cos\theta,$$

$$b = r\sin\theta.$$

Thus,

$$z = a+bi = r\cos\theta + ir\sin\theta.$$

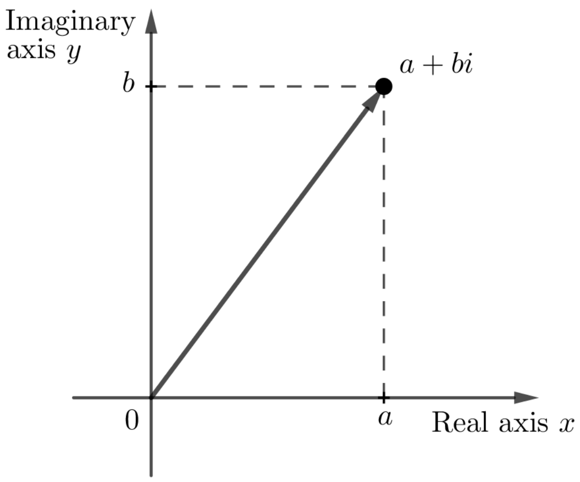

Figure 2.2: Geometrical representation of complex numbers, complex plane

■ **Example 2.7** Let us list a few examples of complex numbers in polar coordinates:

$$i = (1, \frac{\pi}{2}), \ \ -2 = (2,\pi), \ \ 3+\sqrt{3}i = (\sqrt{12}, \frac{\pi}{6}).$$

■

Note that by multiplying $z = a+bi$ and $a-bi$ one obtains

$$(a+bi)\cdot(a-bi) = a^2+b^2$$

which is a nonnegative real number and, by geometric interpretation of complex numbers, it specifies the square of the magnitude of the complex number $a+bi$. Because of this feature, the number $a-bi$ received a special name: *complex conjugate* of $a+bi$. A complex conjugate of a complex number z is denoted by $\bar{z}$. Geometrically, one reflects z across the x axis to obtain $\bar{z}$, see Figure 2.3. Recall that complex conjugates are used to compute quotients of two complex numbers.

R To summarize, a new number set was always introduced to make a "simpler" number set closed with respect to a certain operation. Namely, one obtains $\mathbb{Z}$ from $\mathbb{N}$ by adding new elements to $\mathbb{N}$ to make it closed under subtraction. From $\mathbb{Z}$ one derived $\mathbb{Q}$ by making it closed under the operation of division. By completing $\mathbb{Q}$ by the limits of all rational sequences, $\mathbb{R}$ was derived. Finally, inclusion of all roots of real polynomials in $\mathbb{R}$ led to the definition of $\mathbb{C}$. Hence, there we have the following inclusions:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{Q} \cup \mathbb{I} = \mathbb{R} \subset \mathbb{C}.$$

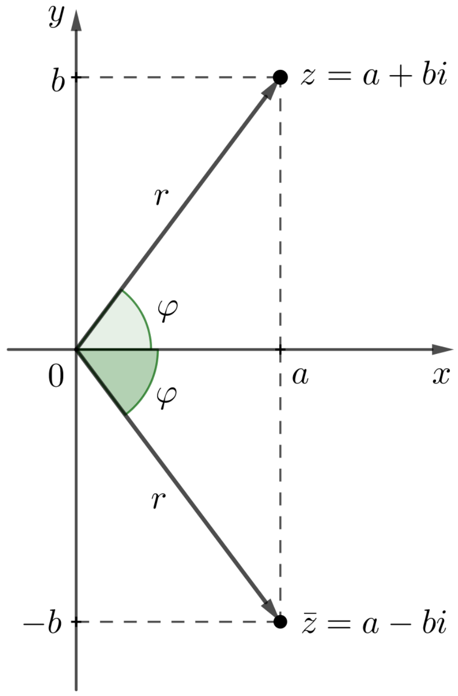

Figure 2.3: Complex conjugate

2.2 Bounds on number sets

On the number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{I}$, $\mathbb{R}$ we have the standard ordering $\leq$. This is the relation which specifies whether a is greater or smaller than b for $a, b \in \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{I}$ or $\mathbb{R}$. In the case when a is smaller or equal to b, we write $a \leq b$. For all these number sets with the standard ordering $\leq$, we can define the concept of a lower and upper bound as well as of a minimum and maximum. Let us state the definitions only in the case of $\mathbb{R}$.

Definition 2.2.1 Let $\emptyset \neq M \subset \mathbb{R}$. We say that the set M is

(i) *bounded from above* if $\exists a \in \mathbb{R}\ \forall m \in M: \ m \leq a$ (a is called an *upper bound* of M),
(ii) *bounded from below* if $\exists a \in \mathbb{R}\ \forall m \in M: \ m \geq a$ (a is called a *lower bound* of M),
(iii) *bounded* if it is bounded from above and bounded from below.

Note that if a is an upper bound of a set S, then each $b \in \mathbb{R}$ such that $b \geq a$ is an upper bound of S as well. Analogously, if a is a lower bound of a set S, then each $b \in \mathbb{R}$ such that $b \leq a$ is a lower bound of S as well.

■ Example 2.8

- The open interval $(1, \infty) = \{x \in \mathbb{R} \,|\, x > 1\}$ is bounded from below. Its lower bounds are e.g. -10, 0.43, 1. The lower bound 1 is called the *greatest lower bound* or *infimum*.
- The interval $(-\infty, 0]$ is bounded from above. Its upper bounds are e.g. 123, 7, 0. The upper bound 0 is called the *least upper bound* or *supremum*.
- The set $\mathbb{Z}$ is neither bounded from above nor bounded from below.
- The set $\{1, 2, 3, 9/2, 4\}$ is bounded, i.e. both bounded from below and bounded from above. Its infimum is 1 and its supremum is $9/2$.

■

Definition 2.2.2 Let $\emptyset \neq M \subset \mathbb{R}$. We say that the number $\max M$ is the *maximum* of M if:

(i) $\forall m \in M: \ m \leq \max M$,

(ii) $\max M \in M$.

We say that the number $\min M$ is the *minimum* of M if:

(i) $\forall m \in M: \; m \geq \min M$,

(ii) $\min M \in M$.

Example 2.9

- The interval $(1,\infty)$ has neither a maximum nor a minimum.
- The maximum of $(-\infty,0]$ is 0. This interval does not have a minimum.
- The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{I}$, $\mathbb{R}$ have neither maxima nor minima.
- The set $\{1,2,3,9/2,4\}$ has a minimum and it equals 1. The maximum of this set is $9/2$.

■

2.3 Cartesian product

In what follows, we introduce the operation of Cartesian product which allows us to deal with ordered pairs in concise way.

Definition 2.3.1 The *Cartesian product* $M \times N$ of sets M and N is the set

$$M \times N = \{(m,n) \,|\, m \in M \wedge n \in N\},$$

that is, it is the set of all ordered pairs (m,n) where $m \in M$ and $n \in N$.

Example 2.10

- Let $M = \{1,2\}$ and $N = \{0,3,4\}$. Then $M \times N = \{(1,0),(1,3),(1,4),(2,0),(2,3),(2,4)\}$.
- For $M = [1,2]$ and $N = \{3\}$, $M \times N = [1,2] \times \{3\} = \{(x,3) \,|\, x \in [1,2]\}$.
- $(-\infty,0) \times (1,\infty) = \{(x,y) \,|\, x < 0 \wedge 1 < y\}$

■

Note that, in general, $M \times N \neq N \times M$, i.e. the operation $\times$ is not commutative.

Example 2.11 Consider $M = \{0\}$ and $N = (-1,1)$. Then

$$M \times N = \{(0,x) \,|\, -1 < x < 1\} \neq \{(x,0) \,|\, -1 < x < 1\} = N \times M.$$

This follows from the fact that

$$\exists\, (m,n) \in M \times N: \quad (m,n) \notin N \times M \quad \text{(for example } (m,n) = (0,1))$$

and

$$\exists\, (n,m) \in N \times M: \quad (n,m) \notin M \times N \quad \text{(for example } (m,n) = (-1,0)).$$

■

Since both the x-axis and the y-axis represent the set of real numbers $\mathbb{R}$, we can write the xy-plane as $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x,y) \,|\, x,y \in \mathbb{R}\}$. Similarly, the xyz-space can be written as $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 = \{(x,y,z) \,|\, x,y,z \in \mathbb{R}\}$. Further, one can graphically represent the Cartesian product of two or three subsets of $\mathbb{R}$ as a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively.

Example 2.12 Consider the set $M \times N = (\{-2\} \cup [1,\infty)) \times ((-\infty,-3) \cup \{2\} \cup [3,\infty))$. The graphical representation of $M \times N$ is the set of points in $\mathbb{R}^2$ such that the x-coordinates of these points lie within $\{-2\} \cup [1,\infty) \subset \mathbb{R}$ and y-coordinates are from $(-\infty,-3) \cup \{2\} \cup [3,\infty) \subset \mathbb{R}$, see Figure 2.4. ■

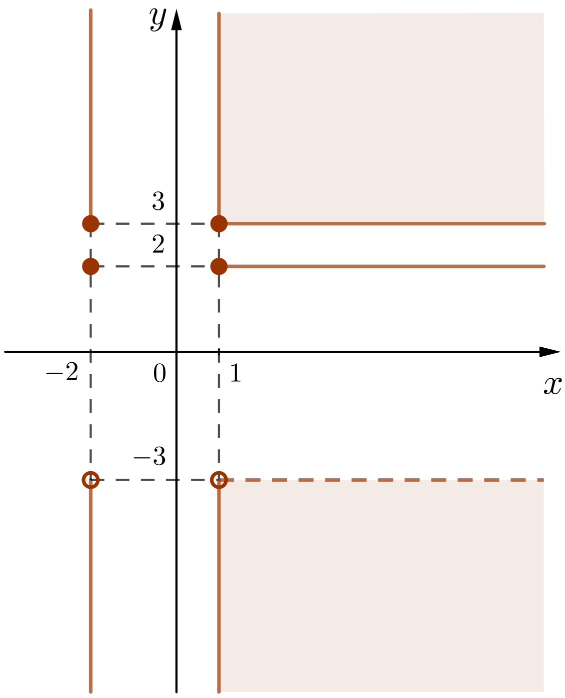

Figure 2.4: $(\{-2\} \cup [1, \infty)) \times ((-\infty, -3) \cup \{2\} \cup [3, \infty))$

2.4 Exercises

Exercise 2.1 Prove that the following number are irrational:
1. $\log_2(3)$,
2. $\log_{\sqrt{2}}(3)$.

■

Exercise 2.2 Are there $a, b \in \mathbb{I}$ such that $a^b \in \mathbb{Q}$? ■

Exercise 2.3 Determine the product of two complex numbers in polar coordinates. Namely, calculate $z_1 \cdot z_2$ for $z_1 = (r_1, \theta_1)$ and $z_2 = (r_2, \theta_2)$.

Hint: Use the trigonometric identities

$$\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a+b)$$

and

$$\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a+b).$$

■

Exercise 2.4 Decide whether the given set A is bounded from below and/or bounded from above. Then decide whether A has a minimum and/or a maximum and if it does, find it.
1. $A = (-3, 5]$
2. $A = \mathbb{N} \cap (-2, 4)$
3. $A = \{1/x \,|\, x \in \mathbb{R} \setminus \{0\}\}$
4. $A = \{x \in \mathbb{Q} \,|\, 1 \leq x^2 < 2\}$

■

Exercise 2.5 Sketch the following sets in $\mathbb{R}^2$:
1. $(-3, 5] \times \mathbb{N}$
2. $(-2, 4) \times ([0, 1) \cup [2, 3])$
3. $\{x \in \mathbb{R} \,|\, x^2 + 2x > 8\} \times \{x \in \mathbb{R} \,|\, \log(x) \leq 2\}$
4. $\{x \in \mathbb{R} \,|\, 2\cos(x) = 1\} \times \{x \in \mathbb{Z} \,|\, 1 \leq x^2 < 20\}$

■

2.5 Answers

Exercise 2.1:

1. Assume for a contradiction that $\log_2(3)$ is rational. Then there are $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $\log_2(3) = \frac{p}{q}$. Because $\log_2(3) > 0$, let us choose $p, q > 0$, i.e. $p, q \in \mathbb{N}$. The equality $\log_2(3) = \frac{p}{q}$ implies that $2^{\frac{p}{q}} = 3$ and thus $2^p = 3^q$. However, the number 2 raised to any positive integer power must be even (because it is divisible by 2) and the number 3 raised to any positive integer power must be odd (since none of its prime factors will be 2). Because an integer cannot be both odd and even at the same time, we have a contradiction and we can conclude that $\log_2(3)$ is irrational.

2. This follows from the previous exercise and from the following computation:

$$\log_{\sqrt{2}}(3) = \frac{\log_2(3)}{\log_2(\sqrt{2})} = \frac{\log_2(3)}{\frac{1}{2}} = 2\log_2(3).$$

Exercise 2.2: Yes, consider, for example, $a = \sqrt{2}$ and $b = \log_{\sqrt{2}}(3)$. Then $a^b = 3$.

Exercise 2.3: $z_1 \cdot z_2 = r_1 \cdot r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) = (r_1r_2, \theta_1+\theta_2)$.

R

- If z_1 and z_2 are negative real numbers, i.e. $\theta_1 = \theta_2 = \pi$, then the argument of the product $z_1 \cdot z_2$ equals $\theta_1 + \theta_2 = 2\pi$ and thus $z_1 \cdot z_2$ is a positive real number.
- By induction, it is possible to derive a general result for the product of n factors in polar form:

$$(r_1,\theta_1)(r_2,\theta_2)\cdots(r_n,\theta_n) = (r_1r_2\cdots r_n, \theta_1+\theta_2+\cdots+\theta_n).$$

As a special case, one obtains the formula for computing the powers of complex numbers in polar form:

$$(r,\theta)^n = (r^n, n\theta).$$

Exercise 2.4:

1. A is bounded, A does not have a minimum, $\max A = 5$.
2. A is bounded, $\min A = 1$, $\max A = 3$.
3. A is neither bounded from below nor bounded from above, A has neither a minimum nor a maximum.
4. A is bounded, $\min A = 1$, A does not have a maximum.

Exercise 2.5: See Figure 2.5

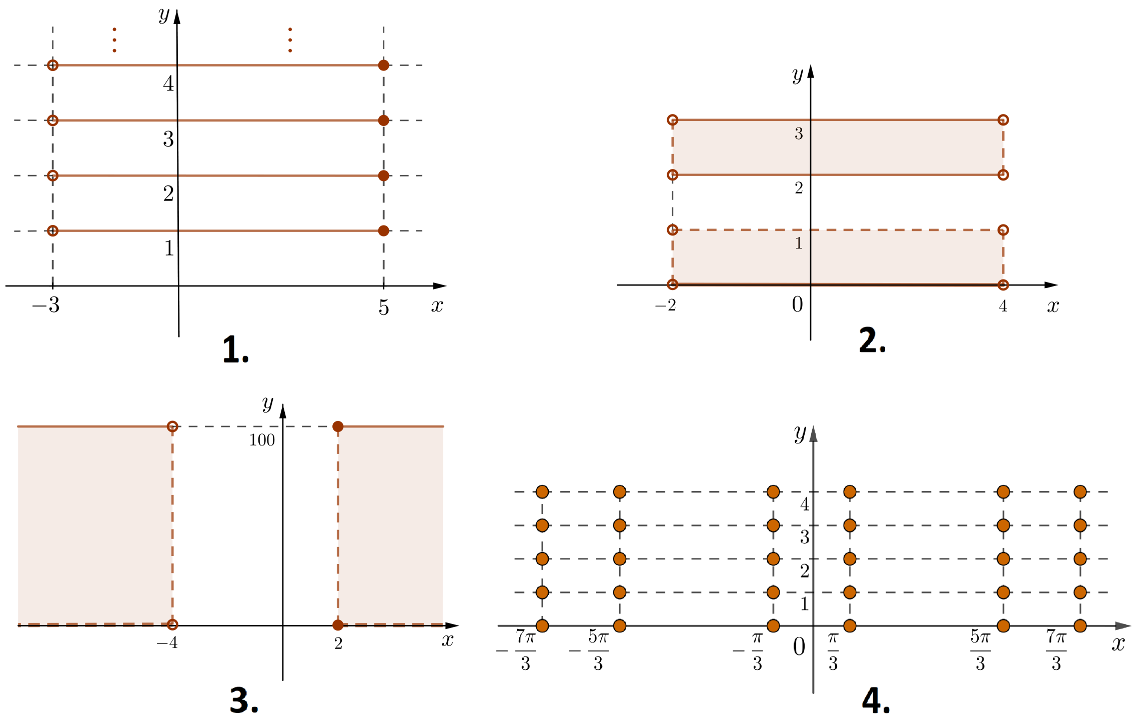

Figure 2.5: Answers to Exercise 2.5