1. Mathematical rigor

The language used within mathematics tries to be as exact and precise as possible. For that the formulation of all statements has to be clear and their validity has to be proved. This language precision and the underlying logic is often referred to as "rigor".

1.1 Formal logic

The reasoning in mathematics is based on a set of rules for making deductions that seem self-evident. The study of these rules is often referred to as formal logic. It deals with statements and deductive arguments with the emphasis on their structure rather than content. One uses symbolic notation which enables easier manipulation and validity testing. Below, several essential notions in formal logic are introduced.

1.1.1 Propositions

By a *proposition* or a *statement* one refers to a meaningful declarative sentence that is either true or false. The truth values "true" and "false" are usually denoted by "1" and "0", respectively. The propositions may be denoted by letters for a reference. These letters are then interpreted as variables representing statements.

■ **Example 1.1** The statement *"Snow is white"* is true. Let us denote this statement by A. Hence, the truth value of A is 1. The statement *"Brno is the capital city of the Czech Republic"*, denoted by B, is false, i.e. the truth value of B is 0. One can say "A holds" and "B does not hold" as well. ■

1.1.2 Logical connectives

Propositions can be combined by *logical connectives/operators* to form more complicated statements. There are five basic logical connectives:

1. *negation* (logical complement) - it reads as "not" and is denoted by $'$ (i.e. A' is the negation of A; other possible notations are $\neg A$ or $NOT\ A$),
2. *conjunction* - it reads as "and" and is denoted by $\wedge$,
3. *disjunction* - it reads as "or" and is denoted by $\vee$,

4. *implication* - it reads as "if (...), then (...)" and is denoted by $\Rightarrow$,
5. *equivalence* - it reads as "if and only if" and is denoted by $\Leftrightarrow$.

The truth values of propositions formed by using logical connectives depend on the truth values of their components. This dependence is summarized in the truth tables below, see Table 1.1. Note that the first row specifies the propositions A, B, and their combinations. The other rows specify the truth values of the respective propositions.

A	A'
1	0
0	1

A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Table 1.1: Truth tables

■ **Example 1.2** Let us discuss the validity of the proposition $(A \Rightarrow B) \Leftrightarrow (B' \Rightarrow A')$, denoted by C, depending on the truth values of propositions A and B.

The goal is to decide when the proposition C is true and when it is false. For that it is sufficient to construct the truth table specifying the truth values of C depending on all possible combinations of the truth values of A and B. One proceeds by gradually filling up larger truth table (column by column) where relevant sub-propositions are considered. These sub-propositions are chosen as relevant combinations of the propositions in the previous columns such that the proposition in the last column is C.

A	B	$D = (A \Rightarrow B)$	B'	A'	$E = (B' \Rightarrow A')$	$C = (D \Leftrightarrow E)$
1	1	1	0	0	1	1
1	0	0	1	0	0	1
0	1	1	0	1	1	1
0	0	1	1	1	1	1

R Note that proposition C is always true. Such a proposition is called a *tautology*.

■

The rules for negation are given in Table 1.2.

Proposition	Negation of proposition
$A \wedge B$	$A' \vee B'$
$A \vee B$	$A' \wedge B'$
$A \Rightarrow B$	$A \wedge B'$
$A \Leftrightarrow B$	$(A \wedge B') \vee (A' \wedge B)$

Table 1.2: Rules for negation

■ **Example 1.3** Let us derive a negation of the proposition $B \Rightarrow [(A \wedge B) \vee C]$ by applying the rules listed in Table 1.2.

$$(B \Rightarrow [(A \wedge B) \vee C])' = B \wedge [(A \wedge B) \vee C]' = B \wedge [(A \wedge B)' \wedge C'] = B \wedge [(A' \vee B') \wedge C']$$

■

■ **Example 1.4** Let us determine the negation C' of the proposition C from Example 1.2. Since C is a tautology, a negation of C has to be a proposition which is always false.

R Propositions which are always false are called *contradictions*.

For example, the proposition $A \wedge A'$ is a contradiction. Thus, $A \wedge A'$ is a negation of C. To derive a negation of C based on the rules listed in Table 1.2 one follows the computation below:

$$\begin{aligned} C' &= [(A \Rightarrow B) \Leftrightarrow (B' \Rightarrow A')]' = [(A \Rightarrow B) \wedge (B' \Rightarrow A')'] \vee [(A \Rightarrow B)' \wedge (B' \Rightarrow A')] \\ &= [(A \Rightarrow B) \wedge (B' \wedge A)] \vee [(A \wedge B') \wedge (B' \Rightarrow A')]. \end{aligned}$$

Note that the proposition $(A \Rightarrow B) \wedge (B' \wedge A)$ is always false since $(B' \wedge A)$ is the negation of $(A \Rightarrow B)$ and a proposition and its negation have never the same truth value. Furthermore, note that the proposition $(A \wedge B') \wedge (B' \Rightarrow A')$ is always false since $(A \wedge B')$ is a negation of $(B' \Rightarrow A')$. So, C' is the disjunction of two propositions which are always false which means that C' is always false. ■

1.1.3 Quantifiers

Propositions may contain *variables* which are simply symbols representing mathematical objects which are either unknown or which may be replaced by any element of a given set, e.g. the set of real numbers. Such propositions may be referred to as *propositional formulas*. Whether a propositional formula is true or false depends on the values assigned to the respective variables.

■ **Example 1.5** *"The number* 3 *is a prime"* is a proposition which is true. Replacing 3 by a symbol x, one obtains the propositional formula *"The number x is a prime"*. This proposition is true, for example, for $x = 3,\ 31,\ 307$ and false, for example, for $x = -3,\ \pi,\ 6$. ■

The variables in propositions may be quantified. There are two fundamental kinds of quantification in mathematical logic: universal quantification and existential quantification. A universal quantification is interpreted as "given any" or "for all". It expresses that a propositional formula can be satisfied by every member of a domain of discourse, i.e. a propositional formula is asserted to be true for every value of a variable. Universal quantification is distinct from existential quantification ("there exists") which asserts that the property or relation holds only for at least one member of the domain. The traditional symbol for the universal quantifier "all" is "$\forall$" and for the existential quantifier "exists" is "$\exists$".

Negation of propositions containing quantified variables follows the rules for negation given in Table 1.2 and the basic rule that the universal quantifier changes to the existential quantifier and vice versa.

■ **Example 1.6** Consider the propositional formula

$$\forall\, x \geq 0\, \forall\, y \geq 0: \ (x+y)^2 \geq x^2 + y^2.$$

It reads as: *For all* $x, y \geq 0$ *it holds that* $(x+y)^2 \geq x^2+y^2$. The negation of this proposition is

$$\exists\, x \geq 0\, \exists\, y \geq 0: \ (x+y)^2 < x^2 + y^2$$

which reads as: *There exist* $x, y \geq 0$ *such that* $(x+y)^2 < x^2+y^2$. ■

1.2 Structure of mathematical texts

Mathematical texts have a fixed structure which makes the exposition simple and clear: the basic statements of the theory which are assumed to hold without any further justification are given in axioms; the objects of interest are named in definitions; the relations and properties of these objects are formulated in theorems and proven in proofs.

1.2.1 Axioms

All mathematical theories are based on a set of *axioms*, mathematical statements from which other statements of the theory are logically derived. Within the system they define, axioms (unless redundant) cannot be derived by principles of deduction nor are they demonstrable by mathematical proofs. The axioms serve as the basic principles, general truths, of the system they define.

As an example, let us state one of the axioms of Zermelo-Fraenkel set theory, the axiom of extensionality:

$$\forall\, x\, \forall\, y:\ (\forall\, z:\ z \in x \Leftrightarrow z \in y) \Rightarrow x = y$$

which says that two sets are equal (are the same set) if they have the same elements.

R The symbol $\in$ denotes the relation of a set membership. It reads as *is an element of*, *in*, *belongs to*, *lies in*. The negation of the set membership relation is denoted by $\notin$.

1.2.2 Definitions and theorems

New terms describing objects or properties are specified in the so-called *definitions*. Definitions give a precise meaning to a new term by using already defined terms.

New statements that have been proven on the basis of previously established statements and axioms are called *theorems* (or propositions, lemmas, and claims depending on the importance of the statements in the theory). Hence, theorems are logical consequences of the axioms. They describe the properties and relations of the objects studied in the theory. The validity of theorems has to be always justified by the proofs following the rules of mathematical logic.

1.2.3 Proofs

Theorems are usually stated in the form of an equivalence or an implication. To prove an equivalence $A \Leftrightarrow B$ one has to prove two implications $A \Rightarrow B$ and $B \Rightarrow A$. Therefore, to discuss the methods for proving theorems in general it is sufficient to discuss the methods for proving an implication $A \Rightarrow B$. There are four basic approaches to prove an implication $A \Rightarrow B$.

Direct proof

One assumes the validity of A and by logical deductions one derives the chain of valid statements $C_1,\, C_2,\, \ldots,\, C_n$ which finally leads to B, i.e. the proof has the structure

$$A \Rightarrow C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_n \Rightarrow B.$$

■ **Example 1.7** Let us prove the statement $\forall\, x \geq 0\, \forall\, y \geq 0:\ (x+y)^2 \geq x^2 + y^2$. One can rewrite it as $\forall\, x\, \forall\, y:\ \ x \geq 0 \wedge y \geq 0 \ \Rightarrow\ (x+y)^2 \geq x^2 + y^2$. Hence, one needs to prove the implication $A \Rightarrow B$, where $A = x \geq 0 \wedge y \geq 0$ and $B = (x+y)^2 \geq x^2 + y^2$.

It holds that $(x+y)^2 = x^2 + 2xy + y^2$. By assuming the validity of A one obtains $2xy \geq 0$ and therefore $(x+y)^2 \geq x^2 + y^2$ which is B and which was to be proven. Written down as a chain of implications, the proof looks as follows:

$$x \geq 0 \wedge y \geq 0 \ \Rightarrow\ 2xy \geq 0 \ \Rightarrow\ 2xy + x^2 + y^2 \geq 0 + x^2 + y^2 \ \Rightarrow\ (x+y)^2 \geq x^2 + y^2.$$

■

Proof by contrapositive

Instead of proving the implication $A \Rightarrow B$, one proves the implication $B' \Rightarrow A'$. The latter implication is then proven directly. Note that the idea of this indirect proof is based on the equivalence $(A \Rightarrow B) \Leftrightarrow (B' \Rightarrow A')$ the validity of which was shown in Example 1.2.

■ **Example 1.8** Let us prove the same statement as in Example 1.7, i.e. $\forall\, x\, \forall\, y: \quad A \Rightarrow B$, where $A = x \geq 0 \wedge y \geq 0$ and $B = (x+y)^2 \geq x^2+y^2$.

Let us assume that $B' = (x+y)^2 < x^2+y^2$ is valid. Then $x^2+2xy+y^2 < x^2+y^2$ and consequently $2xy < 0$. This implies $x < 0 \vee y < 0$ which is the statement of A'. Written down as a chain of implications, the proof looks as follows:

$$(x+y)^2 < x^2+y^2 \quad \Rightarrow \quad x^2+2xy+y^2 < x^2+y^2 \quad \Rightarrow \quad 2xy < 0 \quad \Rightarrow \quad x < 0 \vee y < 0.$$

■

Proof by contradiction

Recall that $(A \Rightarrow B)' \Leftrightarrow (A \wedge B')$. Hence, $A \wedge B'$ is not valid if and only if $(A \Rightarrow B)'$ is not valid and thus $A \Rightarrow B$ holds. Therefore, to prove the implication $A \Rightarrow B$ one can prove that $A \wedge B'$ does not hold. For that, it is sufficient to assume the validity of the statement $A \wedge B'$ and by logical deductions derive a contradiction, i.e. a statement which is not true, for example $C \wedge C'$. Schematically, the proof by contradiction has the following structure:

$$A \wedge B' \Rightarrow \cdots \Rightarrow C \wedge C' \Rightarrow (A \Rightarrow B).$$

■ **Example 1.9** Lastly, let us prove the statement in Example 1.7: $\forall\, x\, \forall\, y: \quad A \Rightarrow B$, where $A = x \geq 0 \wedge y \geq 0$ and $B = (x+y)^2 \geq x^2+y^2$.

Let us assume $A \wedge B'$ holds. Thus, we assume $x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ (x+y)^2 < x^2+y^2$. By rewriting it one obtains $x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ 2xy < 0$ which is a contradiction since the product of two nonnegative numbers has to stay nonnegative. Therefore, the negation of $A \wedge B'$ is valid and, because $(A \wedge B')' = (A \Rightarrow B)$, the statement is proven. ■

Proof by mathematical induction

The statements $A(n)$ which depend on all natural numbers n are most commonly proven by mathematical induction. One proceeds in two steps: 1) *base case*: check the validity of the statement for $n = 1$, i.e. prove $A(1)$, 2) *inductive step*: prove the implication $A(n) \Rightarrow A(n+1)$. Having proved these two steps, the statement $A(n)$ is established to be true for all natural numbers n.

■ **Example 1.10** Let us prove that by adding the first n natural numbers one obtains the sum $\frac{n(n+1)}{2}$. Hence, the statement to be proven is the following:

$$\text{For all natural numbers } n \text{ the statement } A(n) := \quad 1+2+\cdots+n = \frac{n(n+1)}{2} \text{ holds.}$$

Step 1: $A(1)$ holds since $1 = \frac{1(1+1)}{2}$.
Step 2: Let us assume $A(n)$ holds and we prove the validity of $A(n+1)$. $A(n)$ states that $1+2+\cdots+n = \frac{n(n+1)}{2}$. Let us add the number $n+1$ to both sides. Then, $1+2+\cdots+n+(n+1) = \frac{n(n+1)}{2}+(n+1)$. The left-hand side equals $A(n+1)$ and the right-hand side equals $\frac{(n+1)(n+2)}{2}$, which implies the validity of $A(n+1)$. ■

1.3 Exercises

Exercise 1.1 Discuss the validity of the following propositions:

1. $(A \Rightarrow B) \Leftrightarrow (A \wedge B')'$
2. $(A \Leftrightarrow B) \Rightarrow (A \vee B')'$
3. $B' \Leftrightarrow (B \wedge (A' \vee B))$
4. $(A \Rightarrow (A' \Rightarrow B)) \Rightarrow B'$

■

Exercise 1.2 Find the negations of the following propositions:

1. $A \Rightarrow (B \wedge A)$
2. $\exists\, x : x \in A \Leftrightarrow x \in B$
3. $\forall\, x\, \exists\, y : (x \in A \vee x \in B) \Leftrightarrow y \in C$

■

Exercise 1.3 Is the proposition $(A \wedge B)' \wedge ((B \Rightarrow A) \wedge (A \Rightarrow B)) \wedge (A \vee B)$ negation of the proposition $((A \vee B) \Rightarrow (B \wedge A)) \Rightarrow (B \Leftrightarrow A)$? ■

Exercise 1.4 Prove that for any x and y the following holds: $x = y$ if and only if $xy = \frac{(x+y)^2}{4}$. ■

1.4 Answers

Exercise 1.1:

A	B	1.	2.	3.	4.
1	1	1	0	0	0
1	0	1	1	0	1
0	1	1	1	0	0
0	0	1	0	0	1

Exercise 1.2:

1. $A \wedge (B' \vee A')$
2. $\forall\, x : (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)$
3. $\exists\, x\, \forall\, y : (\,(x \in A \vee x \in B) \wedge y \notin C\,) \vee (\,(x \notin A \wedge x \notin B) \wedge y \in C\,)$

Exercise 1.3: Yes

Exercise 1.4: The two implications of the equivalence can be proved directly.