Taylor polynomial. Differential.

Brook Taylor (1685-1731)

Taylor polynomial

Taylor polynomial

Recall: Polynomial of $n-$th degree
$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\,, \quad a_i \in \mathbb{R},\, a_n \neq 0.$

easy to evaluate (Horner's method)

Our goal:
For a given function f and given point x_0 find a "simpler function" such that on a neighborhood of x_0 it has values "close"to $f(x)$. It will be T_n.

Definition: Suppose that function f has n proper derivatives at point x_0, then we define Taylor polynomial of the $n-$th degree of function f at point x_0 as

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

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Remarks:

- It is polynomial of degree at most n.
- $f^{(k)}(x_0)$ are constants.
- $0! = 1, \quad 1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24, \ldots$
- T_1 describes the tangent to the graph.

Assertion: For a Taylor polynomial T_n of function f at point x_0 the following holds true:

$$T_n(x_0) = f(x_0), \quad T_n'(x_0) = f'(x_0), \quad \ldots, \quad T_n^{(n)}(x_0) = f^{(n)}(x_0).$$

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Corollary: In neighborhood of x_0, the Taylor polynomial is a good approximation of function f, i.e.

For $\quad x \doteq x_0 \quad$ we have $\quad f(x) \doteq T_n(x).$

Taylor’s formula

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Theorem: Let f be $(n+1)-$times differentiable on interval I and let T_n be its Taylor polynomial at point $x_0 \in I$, then

$$\forall\, x \in I \quad \exists\, c \in (x, x_0) \text{ (or } c \in (x_0, x) \text{ resp.) such that}$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

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id est

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}\,.$$

The last relation is called the Taylor formula.

Differential of function

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Remarks: Difference Δf $=$ exact change of value.
Differential $\mathrm{d}f$ $=$ approximate change of value.
(approximation by tangent)
The increment Δx can be also negative.