Applications of integrals of functions of one real variable

Riemann definition of definite integral

Definition Let f be continuous on interval $\langle a, b\rangle$.
Let us consider equidistant partition of the interval $\langle a, b\rangle$ with n subintervals

$$a = x_0 < \cdots < x_{n-1} < x_n = b\,, \qquad x_i - x_{i-1} = h.$$

At each subinterval $\langle x_{i-1}, x_i\rangle$ let us choose arbitrary point $c_i \in \langle x_{i-1}, x_i\rangle$. Then the sum

$$S_n(f) = \sum_{i=1}^{n} f(c_i)\cdot(x_i - x_{i-1}) = h\cdot\sum_{i=1}^{n} f(c_i)$$

is called Riemann integral sum.

Riemann integral = $\lim\limits_{n\to\infty} S_n(f)$.

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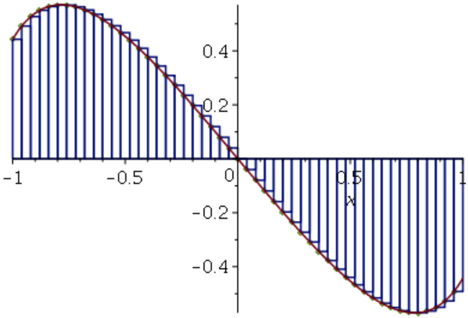
Theorem: Let f be continuous on $\langle a, b\rangle$, then there exists $\lim\limits_{n\to\infty} S_n(f)$ and the limit value does not depend on the choice of points $c_i \in \langle x_{i-1}, x_i\rangle$, $i = 1, \ldots, n$.

Riemann integral



Remark: Here, c_i the left endpoint of the interval

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$$(\mathcal{R}) \int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} S_n(f) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot h.$$

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Theorem: Let f be continuous on interval $\langle a, b\rangle$, then there exists Newton as well as Riemann integrals and their values are equal, id est:

$$(\mathcal{R}) \int_a^b f(x)\mathrm{d}x = (\mathcal{N}) \int_a^b f(x)\mathrm{d}x.$$

Usage:

Riemann integral is useful for deriving general application formulas with integrals.

Newton integral is useful for calculating the ingrals.

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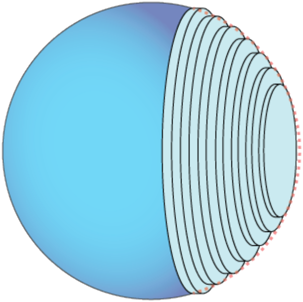
- Volume of solid of revolution, created by revolving a surface bounded by graph of a continuous function f defined on $\langle a, b \rangle$, and lines $x = a$, $x = b$, $y = 0$ around x-axis:

$$V = \pi \int_a^b f^2(x)\,\mathrm{d}x$$

Volume of solid of revolution -idea of the proof



Volume of one disk (cylinder) $\dots S_p \cdot v = \pi r^2 v = \pi f(c_i)^2 h$

Volume of n disks $= \sum_{i=1}^{n}(\pi f(c_i)^2 h) = \pi \sum_{i=1}^{n}(f^2(c_i)h) \doteq$ volume of the solid

volume of the solid
precisely $V = \lim_{n\to\infty} \pi \sum_{i=1}^{n}(f^2(c_i)h) = \pi \int_a^b f^2(x)\,\mathrm{d}x$

$\uparrow$

Riemann sum for f^2

Mean value theorem for definite integrals

Definition: Mean value of continuous function f on interval $\langle a, b \rangle$ is defined as

$$\overline{f} = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \; .$$

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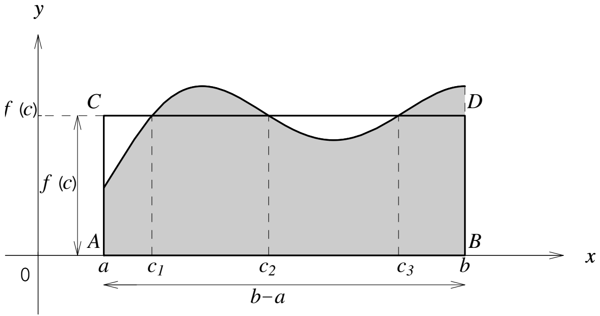
Theorem: Let f be continuous on interval $\langle a, b\rangle$, then existujethere exists $c \in (a, b)$ such that $f(c) = \overline{f}$.



Physical applications

- Work W by non-constant force $\overrightarrow{F}$ acting along a segment $\overline{AB}$, $A = [a; 0]$, $B = [b; 0]$

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$$W = \int_a^b F(x)\,\mathrm{d}x$$

- Work W by a gas enclosed in a cylinder with piston going from position $x = a$ to position $x = b$.

$$W = \int_{V_a}^{V_b} p(V)\,\mathrm{d}V$$