Linear differential equations of 1^{st} and 2^{nd} order with constant coefficients and "special"right hand side

LDE of the 2nd order

Linear differential equation of the 2nd order with constant coefficients is equation of the form

$$k_0 \cdot y''(x) + k_1 \cdot y'(x) + k_2 \cdot y(x) = b(x),$$

where $k_0, k_1, k_2 \in \mathbb{R}$ are constants, $k_0 \neq 0$, and b given function.

Theorem: General solution to the NLDE of the 2nd order can be written in the form

$$y = y_H + y_P$$

where y_H are all solutions of HLDE and
y_p is one arbitrary particular solution to NLDE.

Moreover, the number of linearly independent solutions to the HLDE (the number of integration constants) is equal to the order of the equation (=2).

Structure of the solutions of LDEs

The general approach the same as of the LDE of the first order:

Step 1 Determine all solutions y_H of the corresponding HLDE

$$k_0 \cdot y''(x) + k_1 \cdot y'(x) + k_2 \cdot y(x) = 0$$

Step 2 Determine one arbitrary solution y_p to the original equation

$$k_0 \cdot y''(x) + k_1 \cdot y'(x) + k_2 \cdot y(x) = b(x)$$

Step 3 all solutions $\ldots y_N(x, C_1, C_2) = y_H(x, C_1, C_2) + y_p(x)$.

Step 1 - General solution to the corresponding HLDE

$$k_0 \cdot y''(x) + k_1 \cdot y'(x) + k_2 \cdot y(x) = 0$$

$$\downarrow$$

characteristic equation: $k_0 \lambda^2 + k_1 \lambda + k_2 = 0$

Find the roots $\lambda_{1,2}$ of this quadratic equation.
Consider the three possibilities:

- $D > 0,\ \lambda_1 \neq \lambda_2 \in \mathbb{R} \;\Rightarrow\; y_H(x) = C_1 \mathrm{e}^{\lambda_1 x} + C_2 \mathrm{e}^{\lambda_2 x},$
- $D = 0,\ \lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \;\Rightarrow\; y_H(x) = C_1 \mathrm{e}^{\lambda x} + C_2\, x\, \mathrm{e}^{\lambda x},$
- $D < 0,\ \lambda_{1,2} = a \pm i\, b \in \mathbb{C}$

  $\Rightarrow\; y_H(x) = C_1 \mathrm{e}^{ax} \cos(bx) + C_2 \mathrm{e}^{ax} \sin(bx),$

  (in all cases $x \in \mathbb{R},\ C_1, C_2 \in \mathbb{R}$)

Step 2 - the method of undetermined coefficients

The method is applicable for 1st and 2nd order linear differential equations with

- constant coefficients
  and
- "special" right-hand side
  - has so called finite family of derivatives

Example 1: $y'' + 2y' - y = x^2$
Example 2: $y' + 2y = \sin x$
Example 3: $y'' + y' = x$

For equation

$$a_2\, y'' + a_1\, y' + a_0\, y = \mathrm{e}^{\alpha\, x}\big(P(x)\, \sin(\beta\, x) + Q(x)\, \cos(\beta\, x)\big)$$

there exists a solution y_p of this NLDE in the form

$$y_p(x) = x^k\, \mathrm{e}^{\alpha\, x}(R(x)\, \sin(\beta\, x) + S(x)\, \cos(\beta\, x)), \quad \text{where}$$

- $k \in \{0, 1, 2\}$ is the multiplicity of $\alpha + i\beta$ as a root of characteristic equation, id est

  k=0, if $\alpha + i\beta$ is not a root of characteristic equation

  k=1, if $\alpha + i\beta$ is a single root of characteristic equation

  k=2, if $\alpha + i\beta$ is double root of the characteristic

- R, S - polynomials of degree at most $\max\{\text{dg.}\, P, \text{dg.}\, Q\}$

Thus, we guess roughly the form of y_p. The coefficients of polynomials $R,\ S$ need to be determined so that y_p is a solution to NLDE.

Step 3 General solution to NLDE:

$$y(x) = y_H(x) + y_p(x)$$