Integral calculus

Antiderivative

Definition: let f a function defined on an interval I. Function F defined on I, such that

$$F'(x) = f(x) \quad \forall\, x \in I$$

is called antiderivative / primitive function, or indefinite integral of f on I, we denote

$$F(x) = \int f(x)\,\mathrm{d}x.$$

.

Remark: We cannot expect uniqueness. If F is antiderivative of f on I and $G(x) = F(x) + C$ for $x \in I$, where C is constant, then G is also primitive of f on I.

Existence of antiderivative.

Theorem:
Let f be continuous on interval I, then f has on I antiderivative.

Definite integral

Definition:
Let F be antiderivative of f on interval $\langle a, b \rangle$. The Newton integral of f from a to b is the number $F(b) - F(a)$, we write

$$\int_a^b f(x)\,\mathrm{d}x = \big[F(x)\big]_a^b = F(b) - F(a)\,.$$

Important:
definite integral = number (increment of anitiderivative)
indefinite integral = (primitive) function, antiderivative

Table of antiderivatives

$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1}$ $\quad n \in \mathbb{R},\ n \neq -1, \quad \int \mathrm{d}x = x$

$\int \frac{1}{x} \, \mathrm{d}x = \ln|x|$

$\int \sin x \, \mathrm{d}x = -\cos x$

$\int \cos x \, \mathrm{d}x = \sin x$

$\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a}$ $\quad a > 0,\ a \neq 1$

$\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x$

$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x$

$\int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x$

$\int \frac{\mathrm{d}x}{\sin^2 x} = -\operatorname{cotg} x$

$* \int \frac{f'(x)}{f(x)} = \ln|f(x)|$

Properties of integral

Theorem: Let f and g be continuous on some interval, then it holds

(i) $\displaystyle\int f'(x)\,\mathrm{d}x = f(x) + C$.

(ii) $\displaystyle\left(\int f(x)\,\mathrm{d}x\right)' = f(x)$.

(iii) $\displaystyle\int C\,f(x)\,\mathrm{d}x = C\int f(x)\,\mathrm{d}x$, whenever C is a constant.

(iv) $\displaystyle\int \big(f(x) \pm g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$.

Watch out!

There is NO UNIVERSAL FORMULA for $\displaystyle\int \big(f(x)\cdot g(x)\big)\,\mathrm{d}x$.

Integration by substitution

Substitution method

Theorem: Let $f(t)$ be continuous on $\langle a, b\rangle$ and let $t = \varphi(x)$ have continuous derivative on (α, β) such that $\varphi\big(\langle\alpha, \beta\rangle\big) \subset \langle a, b\rangle$.

Then

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t = F(t) = F\big(\varphi(x)\big),$$

where F is the antiderivative of f on (a, b).

Remark: $t = \varphi(x)$ is the substitution, differential of this function is equal $dt = \varphi'(x)dx$.

Proof: Follows from differentiation of a composed function.

Examples:

$$\int \frac{\sin x}{\cos^2 x}\mathrm{d}x, \quad \int \mathrm{e}^{3x-5}\mathrm{d}x, \quad \int \frac{x}{\sqrt{x^2+1}}\mathrm{d}x$$

Integration by parts

Integration by parts.

Theorem:
Nechť funkce u a v mají v intervalu I spojité derivace. Potom v intervalu I platí

$$\int u(x)\, v'(x)\, \mathrm{d}x = u(x)\, v(x) - \int u'(x)\, v(x)\, \mathrm{d}x\,.$$

Zkráceně:

$$\int u' \cdot v = u \cdot v - \int u \cdot v'$$

Proof: Follows from differentiation of a product.

Usage of integration by parts

1. product of polynomial and sin, cos or exponential e.g..

$$\int x \cdot \sin x \,\mathrm{d}x, \ \int x^3 \cdot \mathrm{e}^{2x} \,\mathrm{d}x, \ \int (x+3) \cdot 2^x \,\mathrm{d}x$$

(we differentiate the polynomial)

2. product of polynomial (or x^{-n}) and inverse trigonometric or logarithmic function, or their powers e.g.

$$\int \arcsin x \,\mathrm{d}x, \ \int x^{-2} \cdot \ln x \,\mathrm{d}x, \ \int (\log x)^2 \,\mathrm{d}x$$

(we integrate the polynomial)

3. product of trigonometric and exponential function e.g.

$$\int \mathrm{e}^{2x} \cdot \sin x \,\mathrm{d}x, \ \int 2^x \cdot \cos 3x \,\mathrm{d}x$$

(leads to an equation)

Rational functions.

Definition: A function $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$, $Q(x)$ are polynomials, is called a rational function.
Whenever degree of $P(x) <$ degree of $Q(x)$, it is called proper rational function.
Otherwise it is called improper rational function.

Theorem: Every rational function can be written as a sum of polynomial and a proper rational function.

(Use the long division of polynomials.)

Remark: We know how to integrate polynomial. We aim at integrating proper rational function.

We will write a proper rational function as a sum of so-called partial fractions (functions which can be integrated by previous methods) .

Integration of rational functions

1. převod rac. lomené funkce na součet polynomu a ryze lomené funkce (dělením polynomů)
2. rozklad jmenovatele na součin kořenových činitelů
3. rozklad **ryze** lomené racionální funkce na součet tzv. parciálních zlomků
4. integrace parciálních zlomků

Rozklad polynomu na kořenové činitele

Definition - recall: Polynomial of degree n...
$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
$a_0, a_1, \ldots, a_n \in \mathbb{R}$ are coefficients and $a_n \neq 0$

Root of polynomial $P_n(x)$ is a number α such that $P_n(\alpha) = 0$.

Our aim = polynomial factorization

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 =$$
$$= a_n (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k) \cdot (x^2 + p_1 x + q_1) \cdots (x^2 + p_\ell x + q_\ell)$$

- α_i are all roots of $P(x)$
- $(x^2 + p_j x + q_j)$ do not have real roots, i.e. $D_j < 0$.
  $(x^2 + p_j x + q_j) = (x - \alpha_1)(x - \alpha_2)$, where $\alpha_{1,2}$ are complex roots of $x^2 + p_j x + q_j$.

Polynomial factorization (2)

Theorem: If α is a root of polynomial $P_n(x)$, then

$$P_n(x) = (x - \alpha)Q(x)\,,$$

where $Q(x)$ is polynomial of degree $(n - 1)$.

Definition: We say that α is a root of $P(x)$ of multiplicity k, if

$$P(x) = (x - \alpha)^k Q(x),$$

where $Q(x)$ is polynomial and $Q(\alpha) \neq 0$.

Note: If α is a is a root of $P(x)$ of multiplicity $k \Leftrightarrow$

$$P(\alpha) = 0,\; P'(\alpha) = 0, \ldots, P^{k-1}(\alpha) = 0 \text{ a } P^k(\alpha) \neq 0.$$

Rozklad polynomu na kořenové činitele (3)

Fundamental theorme of algebra : Every polynomial of degree $n \geq 1$has, counting with multiplicity, exactly n roots, (some roots can be complex).

Partial fractions decomposition.

1. Factorize the polynomial in denominator.
2. If a linear factor $(a\,x + b)$ appears in the factorization, then in the partial fraction decomposition appears

$$\cdots + \frac{A}{(a\,x + b)} + \cdots$$

where A is some real constant.

3. If a term $(a\,x + b)^k$, $k = 2, 3, \ldots$, appears in the factorization, then in the partial fraction decomposition appears

$$\cdots + \frac{A_1}{(a\,x + b)} + \frac{A_2}{(a\,x + b)^2} + \cdots + \frac{A_k}{(a\,x + b)^k} + \cdots$$

with $A_1,\ A_2, \ldots, A_k$ some suitable constants.

Partial fractions decomposition.

4. If $(a x^2 + b x + c)$ (with $D < 0$) appears in the factorization, then in the partial fraction decomposition appears

$$\cdots + \frac{A x + B}{(a x^2 + b x + c)} + \cdots$$

where $A, \; B$ are some suitable constants.

Integration of partial fractions

1. $\int \frac{A}{a\,x+b}\,\mathrm{d}x = \frac{A}{a}\ln|a\,x+b|$ (substitution $t = a\,x + b$)
2. $\int \frac{A}{(a\,x+b)^k}\,\mathrm{d}x = -\frac{A}{a(k-1)}\frac{1}{(a\,x+b)^{k-1}}$ (substitution $t = a\,x + b$)
3. $\int \frac{A\,x+B}{(a\,x^2+b\,x+c)}\mathrm{d}x = \int \frac{A_1\,(2a\,x+b)}{(a\,x^2+b\,x+c)}\mathrm{d}x + \int \frac{B_1}{(a\,x^2+b\,x+c)}\mathrm{d}x$

ad (3) $A_1,\ B_1$ are new constants. For the first integral the substitution $t = a\,x^2 + b\,x + c$, can be used note that $\mathrm{d}t = (2a\,x + b)\mathrm{d}x$. The second one leads to arctg **.**