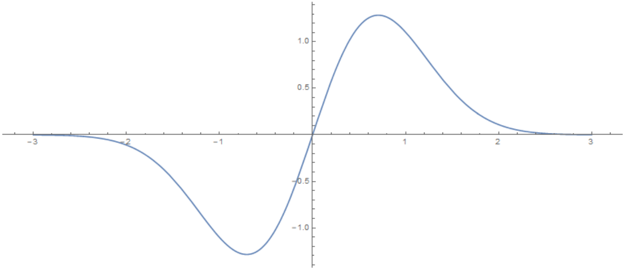
Graphing plots of functions



Examining the monotony

Theorem:
Let f be continuous and differentiable function on interval I. Then

(i) if $f'(x) > 0$ on I, then f is increasing on I.
(ii) if $f'(x) \geq 0$ on I, then f is non-decreasing on I.
(iii) if $f'(x) < 0$ on I, then f is decreasing on I.
(iv) if $f'(x) \leq 0$ on I, then f non-increasing on I.
(v) if $f'(x) = 0$ on I, then f is constant on I.

Be careful: The assertion holds for intervals only!
E.g. for $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2} < 0$ on $(-\infty, 0) \cup (0, \infty)$,
but f is not increasing on ~~$(-\infty, 0) \cup (0, \infty)$~~.

Corollary:
We determine the intervals, where f increases or decreases, respectively (so called intervals of monotonicity).

Local extrema

Definition: We say that function f has at the point x_0 local maximum, if $\exists\, \mathcal{P}(x_0)$ such that
$$\forall\, x \in \mathcal{P}(x_0):\ f(x) \leq f(x_0).$$

Analogically: We say that function f has at the point x_0 local minimum, if $\exists\, \mathcal{P}(x_0)$ such that
$$\forall\, x \in \mathcal{P}(x_0):\ f(x) \geq f(x_0).$$

Remark: If a strict inequality is satisfied we talk about strict local maximum (or minimum).

Local extrema ... common name for local minimums and maximums.

Finding local extrema

Theorem: Suppose that f is continuous on (a, b) and $x_0 \in (a, b)$.

(i) If $f'(x) > 0$ on (a, x_0) and $f'(x) < 0$ on (x_0, b), then f has at point x_0 a strict local maximum.

(ii) If $f'(x) < 0$ on (a, x_0) and $f'(x) > 0$ on (x_0, b), then f has at point x_0 a strict local minimum.

(iii) If $f'(x_0) \neq 0$, then f does not have a local extreme at x_0.

points "suspicious" of being local extrema $\begin{cases} f'(x_0) = 0 \; \ldots \text{ stationary points} \\ f'(x_0) \text{ does not exist} \end{cases}$

Global extrema

Definition: We say that function f has a global maximum at point $x_0 \in D(f)$, if

$$\forall x \in D(f) \;:\; f(x) \leq f(x_0)\,.$$

Analogically: We say that function f has a global minimum at point $x_0 \in D(f)$, if

$$\forall x \in D(f) \;:\; f(x) \geq f(x_0)\,.$$

We refer to the number $f(x_0)$ as maximal (or minimal) value of function f.

Remark: Not all function possess maximal and minimal value.

Theorem: Let f be function continuous on $\langle a, b\rangle$, then f attains its maximal and minimal value on $\langle a, b\rangle$.

Remark: Maximal and minimal values can be attained either at local extrema or at the endpoints of the domain of definition.

Convex and concave functions

Definition: Let f be function defined on interval I.

1. If for any triple $x_1 < x_2 < x_3$, $x_1,\ x_2,\ x_3 \in I$, point $P_2 = [x_2, f(x_2)]$ lies below or on the line connecting points $P_1 = [x_1, f(x_1)]$ and $P_3 = [x_3, f(x_3)]$, we say that the function is convex on I.
2. If point P_2 always lies above or on the line, we say that the function is concave on I.

Theorem:
Let function f be twice differentiable on interval I. Then it holds:

(i) If $f''(x) \geq 0$ on I, then f is convex on I.

(ii) If $f''(x) \leq 0$ on I, then f is concave on I.

Theorem : Let f be defined on (a, b) and $x_0 \in (a, b)$.
If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has local minimum at point x_0.
If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has local maximum at point

Inflection points

Definition: Let f be continuous on (a, b), $x_0 \in (a, b)$ and let $f'(x_0)$ exist (proper or improper). If f is convex on (a, x_0) and concave on (x_0, b) (or vice versa), then we say that f has inflection at point x_0 or that graph of function f has at point $[x_0, f(x_0)]$ inflection point.

Theorem: Let f be twice differentiable function on interval (a, b).
If $f''(x) > 0$ on (a, x_0) and $f''(x) < 0$ on (x_0, b) (or vice versa), then f has inflection at point x_0.

Asymptots

Definition: If

$$\lim_{x\to a+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a-} f(x) = \pm\infty$$

then the line $x = a$ is called vertical asymptote of the graph of f.

Definition: If

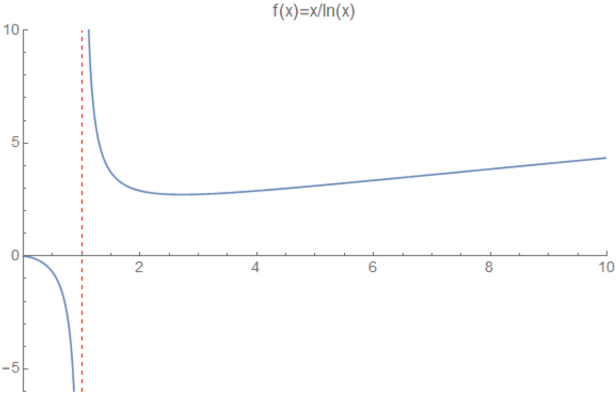
$$\lim_{x\to\infty} f(x) = b \in \mathbb{R} \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b \in \mathbb{R}$$

then the line $y = b$ is called horizontal asymptote of the graph of f at ∞ (or $-\infty$ respectively).

Graphing function

1. Determine $D(f)$
2. Parity, periodicity.
3. *Intersection points with axes, if possible.*
4. Continuity, (one-sided) limits at endpoints of $D(f)$ and at the points of discontinuity $\Rightarrow$ vertical and horizonatl asymptots
5. $f'(x) \Rightarrow$ monotonicity, local extrema.
6. $f''(x) \Rightarrow$ convexity, concavity, inflection.
7. Sketch the graph of f a determine $Im(f)$.

$$f(x) = \frac{x}{\ln x}$$



Numerical solution of equation $f(x) = 0$.

Recall Root of equation is such number α, for which the equation is satisfied (here $f(\alpha) = 0$).

1. determine the number of roots (e.g. by graph)
2. for each root determine the so-called separation interval

   Definition: Suppose that in interval $\langle a, b\rangle$ there exists exactly one root of equation, then we call $\langle a, b\rangle$ a separation interval.

   Theorem:
   Let f be continuous on $\langle a, b\rangle$ and let $f(a) \cdot f(b) < 0$, then there is at least one root of equation $f(x) = 0$ in interval (a, b) .
   If additionally $f'(x) \neq 0$ for all $x \in \langle a, b\rangle$, then there is exactly one root of equation $f(x) = 0$ in interval (a, b) .
3. find approximately root α - lot of root-finding methods - e.g.
   - bisection method . . . simple naive method
   - secant method
   - by Newton method

     . . . numerical method for approximating the solution

Newton method (method of tangents)

Let f continuous twice differentiable function on (a, b) and let α be a root of equation $f(x) = 0$ in interval $\langle a, b\rangle$

Choose $x_0 \in \{a, b\}$ such that $f(x_0) \cdot f''(x_0) > 0$.
construct sequence $\{x_n\}_{n=0}^{\infty}$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$\{x_n\}_{n=0}^{\infty}$ . . . sequence of approximations of root α.
x_n . . . $n-$th approximation of root α.

Theorem: Let f be continuous on $\langle a, b\rangle$. If the following assumptions hold

(i) $f(a) \cdot f(b) < 0$

(ii) $f'(x) \neq 0$ for all $\forall\, x \in \langle a, b\rangle$

(iii) $f''(x) \neq 0$ for all $\forall\, x \in \langle a, b\rangle$

(iv) $x_0 \in \{a, b\},\ f(x_0) \cdot f''(x_0) > 0$,

then the Newton method converges, i.e. for the sequence $\{x_n\}_{n=0}^{\infty}$ it holds $\lim\limits_{n\to\infty} x_n = \alpha$.