Real functions

Function of one real variable

Definition:
Suppose $M \subset \mathbb{R}$. If we assign for each $x \in M$ **uniquely** by a mapping f some $y \in \mathbb{R}$, we say that y is function of x.

x ... independent variable (input)
y ... dependent variable (output)

$M = D(f)$ ... domain of definition f
$Im(f) = H(f) = \{y \in \mathbb{R} | y = f(x), x \in D(f)\}$ ... range, image f

Definition: $\mathrm{graph}(f) = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in D(f)\}$

Graph f is a set of ordered pairs $(x, f(x))$, set of points in a plane.

Examples of graphs

Price of Phillip Morris shares during 2002

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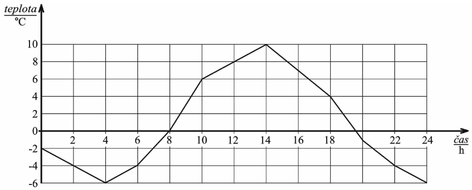
Measured temperature on a given place during 24 hours



Functions can be specified

- by formula
- by graph
- table, algorithm, ...

The domain of the definition is an integral part of definition of the function. If it is not specified, we consider the so called natural domain of definition.

Table I - crucial

For all functions you need to know $D(f)$, $Im(f)$, distinguished values and limits (we will see later)!!!

HW - Table I

Operations with functions

- Sum and difference of functions $h = f \pm g$:
  $h(x) = f(x) \pm g(x)$
- Product of functions $h = f \cdot g$: $h(x) = f(x) \cdot g(x)$
- Quotient of functions $h = \frac{f}{g}$: $h(x) = \frac{f(x)}{g(x)}$
- Composition of functions $h = g \circ f$:
  $h(x) = (g \circ f)(x) = g\big(f(x)\big)$

  g - outer function, f - inner function

  Remark: Generally $g \circ f \neq f \circ g$. (e.g. $\cos^2(x) \neq \cos(x^2)$)

Properties of functions - injectivity

Definition: Function f is injective on $M \subseteq D(f)$, whenever for each pair $x_1, x_2 \in M$ it holds

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

Remark: We say that f is injective, if it is injective on $D(f)$.

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Remark:

- Equivalent formulation to prove that f is injective

$$\forall x_1, x_2 \in D(f) : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

- negation to prove that f is not injective

$$\exists x_1, x_2 \in D(f) : x_1 \neq x_2 \wedge f(x_1) = f(x_2)$$

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- **Theorem:** The composition of injective functions is injective.

Be careful! Function $f(x) = \begin{cases} e^x, & x \in (-\infty, 0\rangle \\ \sqrt{x}, & x \in (0, \infty) \end{cases}$ is not injective.

Monotony of functions

Definice: Let f be a function and $M \subseteq D(f)$. If for all $x_1, x_2 \in M$, $x_1 < x_2$ it holds

(i) $f(x_1) < f(x_2)$, then f is increasing on M

(ii) $f(x_1) > f(x_2)$, then f is decreasing on M

(iii) $f(x_1) \leq f(x_2)$, then f is non-decreasing on M

(iv) $f(x_1) \geq f(x_2)$, then f is non-increasing on M

If f has one of the properties $(i) - (iv)$, it is said to be monotone. If f is from (i) or (ii), we call it strictly monotone.

Remark We say that f is increasing(decreasing, ...), if it is increasing (decreasing, ...) on its $D(f)$.

Bounded functions

Definition:

- We say that f is bounded from below, iff

$$\exists b \in \mathbb{R} \text{ such that } \forall x \in D(f) \text{ it holds } b \leq f(x).$$

- We say that f is bounded from above, if

$$\exists a \in \mathbb{R} \text{ such that } \forall x \in D(f) \text{ it holds } f(x) \leq a.$$

- Function is said to be bounded, if it is bounded from below and from above.

Parity of functions

Definition:

- We say that f is even, whenever
  (i) $x \in D(f) \Leftrightarrow -x \in D(f)$
  (ii) $\forall x \in D(f): \; f(-x) = f(x)$
- We say that f is odd, whenever
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Remarks:

(i) The domain of definition of odd or even function need to be symmetrical around zero.

(ii) The graph of even function is axially symmetrical with axis y.

(iii) The graph of odd function has point symmetry with respect to the origin $[0, 0]$.

Periodic function

Definition: A function f is said to be periodic, whenever $p > 0$ such that:

(i) $x \in D(f) \Rightarrow x \pm p \in D(f)$

(ii) $\forall x \in D(f): \ f(x \pm p) = f(x)$

The smallest such p is called the fundamental period.

Functions $\sin x$ a $\cos x$ are 2π-periodic,
functions $\operatorname{tg} x$ a $\operatorname{cotg} x$ are π-periodic.

Inverse function

Definition: Let f be a given injective function with range $Im(f)$, then there exists function f^{-1} such that

- $D(f^{-1}) = Im(f)$
- $y = f(x) \Leftrightarrow x = f^{-1}(y)$.

Function f^{-1} is called inverse function of f.

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It holds:

(i) Graphs of f and f^{-1} are mutually symmetrical with respect to line $y = x$.

(ii) $Im(f^{-1}) = D(f)$

(iii) $\forall x \in D(f) : f^{-1}\big(f(x)\big) = x$

(iv) $\forall y \in D(f^{-1}) : f\big(f^{-1}(y)\big) = y$

(v) $\left(f^{-1}\right)^{-1} = f$

(vi) $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Examples of inverse functions

- **Powers and roots**

$$f(x) = x^3 \qquad \Rightarrow \qquad f^{-1}(x) = \sqrt[3]{x}$$

$$f(x) = x^2, \quad x \geq 0 \qquad \Rightarrow \qquad f^{-1}(x) = \sqrt{x}$$

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- **Exponentials and logarithms**

$$y = a^x \ \Leftrightarrow\ x = \log_a(y), \quad x \in \mathbb{R}, y > 0$$

Useful: $h(x) = f(x)^{g(x)} = \left(e^{\ln(f(x))}\right)^{g(x)} = e^{g(x)\cdot\ln(f(x))}$

"Inverse trigonometric" functions

Definition:

$$f(x) = \sin x, \quad x \in \langle -\tfrac{\pi}{2}, \tfrac{\pi}{2} \rangle \quad \Longrightarrow \quad f^{-1}(x) = \arcsin(x)$$

$$f(x) = \cos x, \quad x \in \langle 0, \pi \rangle \quad \Longrightarrow \quad f^{-1}(x) = \arccos(x)$$

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Theorem:

$f(x)$	$\arcsin(x)$	$\arccos(x)$	$\operatorname{arctg}(x)$	$\operatorname{arccotg}(x)$
$D(f)$	$\langle -1, 1 \rangle$	$\langle -1, 1 \rangle$	$\mathbb{R}$	$\mathbb{R}$
$H(f)$	$\langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$	$\langle 0, \pi \rangle$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(0, \pi)$
rostoucí	$\checkmark$	$-$	$\checkmark$	$-$
klesající	$-$	$\checkmark$	$-$	$\checkmark$
sudá	$-$	$-$	$-$	$-$
lichá	$\checkmark$	$-$	$\checkmark$	$-$
$f^{-1}(x)$	$\sin(x)$	$\cos(x)$	$\operatorname{tg}(x)$	$\operatorname{cotg}(x)$
	$x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$	$x \in \langle 0, \pi \rangle$	$x \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$x \in (0, \pi)$