Differential equations

What is differential equation?

Definition: Differential equation is a relation between a function $y(x)$ (which is the unknown), its derivatives $y'(x), y''(x), y'''(x), \ldots$ and independent variable x.

E.g.:

$$y''' - x\,y'' - 2\,y\,y' \;=\; x^2\,sinx$$

Definition: Solution of a differential equation is any function $y(x)$, $x \in I$ (defined on interval!) such that $\forall x \in I$ the equation is satisfied.

Definition:
general solution . . . set of all solutions
particular solution . . . one specific solution
integrální křivka . . . graph of a particular solution
order of the equation . . . order of the highest derivative in the equation

Equations of the first order

We will be able to solve two kinds of DE's of the first order:

1. Separable DE

$$y' = f(x) \cdot g(y)$$

2. Linear DE of the first order

$$a_0(x)y' + a_1(x)y = b_1(x), \quad a_0(x) \neq 0$$

$$\text{resp.} \quad y' + a(x)y = b(x)$$

Equations $y' = f(x) \cdot g(y)$.

Theorem: (Existence and uniqueness) (without proof)
Consider the equation $y' = f(x) \cdot g(y)$.
Let $f(x)$ be continuous on interval (a, b) and $g'(y)$ continuous on interval (c, d), then for every point of rectangle $\mathcal{O} = (a, b) \times (c, d)$ there is exactly one integral curve passing through this point.
Id est, for every point $[x_0, y_0] \in \mathcal{O}$ there exists unique solution of equation $y' = f(x)\, g(y)$ satisfying the initial condition $y(x_0) = y_0$.

Geometrical meaning: To every point $[x, y]$ in the rectangle $\mathcal{O}$ we assign a small line segment with directrix $k = f(x) \cdot g(y)$, obtaining directional field. The line segment is tangent to the integral curve passing through $[x, y]$.

Example: $y' = -\frac{x}{y}$ on $\mathcal{O} = (-\infty, \infty) \times (0, \infty)$

Separation of variables for $y' = f(x)\,g(y)$.

1. Find out all y_0, for which $g(y_0) = 0$.
   Then $y \equiv y_0$ is a constant solution of DE $y' = f(x)\,g(y)$.

2. $$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y),$$

3. Separate the variables : $$\frac{\mathrm{d}y}{g(y)} = f(x)\mathrm{d}x$$

4. $$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\mathrm{d}x$$

5. Calculate the antiderivatives:
   $$G(y) + C_1 = F(x) + C_2, \qquad \text{where } C_1, C_2 \in \mathbb{R} \text{ constants}$$
   $$G(y) = F(x) + C, \qquad \text{with } C = C_2 - C_2.$$

6. Extract y: $y(x) = G^{-1}(F(x) + C),$ where $x \in I$.

Separation of variables-summary

Věta: Let $f(x)$ be continuous on (a, b) and $g'(y)$ continuous on (c, d). Then:

1. If $g(y_0) = 0$ for some $y_0 \in (c, d)$, then the constant function
$$y(x) \equiv y_0$$
defined on (a, b) is solution to the equation $y' = f(x)\, g(y)$.

2. If $g(y) \neq 0$ for all $y \in (c, d)$, then the general solution to the equation $y' = f(x)\, g(y)$ on rectangle $(a, b) \times (c, d)$ reads as
$$y(x) \ = \ G^{-1}\big(F(x) + C\big),$$
where $F(x) = \displaystyle\int f(x) \mathrm{d}x$ a $G(y) = \displaystyle\int \frac{1}{g(y)} \mathrm{d}y$.

Linear differential equation of the first order

Let us assume that $a_0(x)$, $a_1(x)$, $b_1(x)$, $a(x)$, $b(x)$ are continuous functions on $I = (\alpha, \beta)$

Definition:
Linear differential equation of the first order is an equation in form

$$a_0(x)y' + a_1(x)y = b_1(x), \quad a_0(x) \neq 0 \ \forall x \in I$$

or

$$y' + a(x)y = b(x).$$

- $$y' + a(x)y = 0$$
  is called homogeneous linear differential equation (HLDE) of the first order.
- $$y' + a(x)y = b(x), \quad \exists x \in I : b(x) \neq 0$$
  is called non-homogeneous linear differential equation (NLDE) of the first order.

Existence and uniqueness for LDE

Theorem: (without proof)
Consider equation

$$y' + a(x)y = b(x)\,.$$

Let $a(x), b(x)$ be continuous on $I = (a, b)$, then for any $[x_0, y_0] \in \mathcal{O} = I \times \mathbb{R}$ there exists unique solution of equation $y' + a(x)y = b(x)$ satisfying the initial condition $y(x_0) = y_0$.

Remark 1: The same for

$$a_0(x)y' + a_1(x)y = b_1(x), \quad a_0(x) \neq 0 \; \forall x \in I.$$

Remark 2: The domain of definition of the solution to LDE is always whole interval I.

Structure of solutions to LDE

Theorem: General solution to NLDE of the first order

$$y' + a(x)y = b(x)$$

is in the form

$$y = y_H + y_p$$

where y_H are all solutions to HLDE and

y_p is one arbitrary particular solution to NLDE.

Theorem: The general solution to HLDE of the first order

$$y' + a(x)y = 0$$

can be written in the form

$$y_H(x) = C\,\mathrm{e}^{-A(x)},\ C \in \mathbb{R},$$

where $A(x) = \int a(x)\mathrm{d}x$.

Remark: You don't have to memorize the formula for y_H, (separation).

Solution NLDE - variation of constant.

Consider NLDE

$$y' + a(x)y = b(x), \quad x \in I.$$

1. Solution of corresponding HLDE $y' + a(x)y = 0$
   y_H is in the form: $y_H(x) = C \cdot \varphi(x), \; C \in \mathbb{R}$

   (možno použít vzorec $y_H(x) = C\, \mathrm{e}^{-A(x)}$)

2. **Variation of constant** = Look for y_p in the form

   $$y_p(x) = c(x) \cdot \varphi(x),$$

   where $c(x)$ is some function on I.

   • Plug y_p into NLDE, to get equation for $c'(x)$

   $$c'(x)\varphi(x) = b(x).$$

   • Calculate $c'(x)$ and integrate to obtain $c(x) = \int \frac{b(x)}{\varphi(x)} \mathrm{d}x$.

3. Všechna řešení NLDR tedy jsou:

   $$y = y_p + y_H = C \cdot \varphi(x) + c(x) \cdot \varphi(x), \; C \in \mathbb{R}$$

Variation of constant var. 2 - summary

Věta: Consider NLDE

$$a_0(x)y' + a_1(x)y = b_1(x), \quad a_0(x) \neq 0 \quad \forall x \in I$$

and a solution to corresponding HLDE in the form

$$y_H(x) = C \cdot \varphi(x).$$

Whenever $c(x)$ satisfies

$$c'(x)\varphi(x) = \frac{b_1(x)}{a_0(x)},$$

then the function $y_p(x) = c(x)\varphi(x)$ is a solution to the NLDE.

Example 1: $y' - \frac{1}{2\sqrt{x}}y = e^{\sqrt{x}}$

Example 2: $y' - \frac{2}{x}y = x^3, \quad y(1) = \frac{3}{2}$

Euler method

Numerical method for approximating a solution of the initial value problem:

$$y' = f(x, y) \qquad y(x_0) = y_0.$$

After n steps we will obtain an aproximative value of the solution at points $x_0, x_1, \ldots, x_n$.
Let us denote the approximative values as $y_i \doteq y(x_i)$.

The approximative values are calculated using the Euler method as follows:

$$\begin{aligned} x_{i+1} &= x_i + h, \\ y_{i+1} &= y_i + f(x_i, y_i) \cdot h \qquad i = 0, 1, \ldots, n \end{aligned}$$

where h is the step of the method. The error $E(h) = y_n - y(x_n)$ is directly proportional to $h \, (= h^1)$. We say that the Euler method is a method of the first order.