# Derivatives

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**Remark:** By substitution $x - x_0 = h$, we get equivalent formula

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

# Geometrical meaning of derivative

**Theorem: (Geometrical meaning) !!!**
Derivative $f'(x_0)$ je slope of the tangent line to the graph of function $f$ at point $[x_0, f(x_0)]$

$\rightsquigarrow$ **Consequence:** equation of the tangent line $y - f(x_0) = f'(x_0)(x - x_0)$

# Importance of derivative in applications

- **physics:** $\rightsquigarrow$ instantaneous velocity $v(t) = s'(t)$

$$\frac{s(t_0 + \triangle t) - s(t_0)}{\triangle t} \quad \underset{\triangle t \to 0}{\longrightarrow} \quad s'(t_0)$$

average velocity within time interval $\langle t_0, t_0 + \triangle t \rangle$ — instantaneous velocity at time $t_0$

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- **chemistry:** $\rightsquigarrow$ rate of change for reaction $w(t) = c'(t)$
instantaneous change of concentration

$$\frac{c(t_0 + \triangle t) - c(t_0)}{\triangle t} \quad \xrightarrow[\triangle t \to 0]{} \quad c'(t_0)$$

average rate of change within the time interval $\langle t_0, t_0 + \triangle t \rangle$ — instantaneous chemical reaction rate at time $t_0$

# One-sided derivatives at a point

**Definition**:

- Right-hand derivative of $f$ at point $x_0$

$$f'_+(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0}$$

- Left-hand derivative of $f$ at point $x_0$

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**Theorem:** Function $f$ is differentiable at point $x_0$ if and only if $f'_+(x_0) = f'_-(x_0)$. Then it holds:

$$f'(x_0) = f'_+(x_0) = f'_-(x_0)$$

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- Function $f$ has derivative on interval $(a, b)$ $\Leftrightarrow$ $f$ has derivative at each point of $(a, b)$.
- Function $f$ has derivative on interval $\langle a, b\rangle$ $\Leftrightarrow$ $f$ has derivative on $(a, b)$ and it has one-sided derivatives $f'_+(a)$, $f'_-(b)$.

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**Theorem:**

If $f$ has proper derivative on interval $I$, then $f$ is continuous on $I$.

(! the reverse implication is not true !)

# Derivatives of elementary functions

$$(k)' = 0 \qquad k \in \mathbb{R}$$

$$(x^a)' = a \cdot x^{a-1} \qquad a \in \mathbb{R}$$

$$(a^x)' = a^x \cdot \ln(a) \qquad 1 \neq a > 0$$

$$(\log_a(x))' = \frac{1}{x \cdot \ln(a)} \qquad 1 \neq a > 0$$

$$(\sin(x))' = \cos(x) \qquad (\cos(x))' = -\sin(x)$$

$$(\mathrm{tg}(x))' = \frac{1}{\cos^2(x)} \qquad (\mathrm{cotg}(x))' = \frac{-1}{\sin^2(x)}$$

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos(x))' = \frac{-1}{\sqrt{1-x^2}}$$

$$(\mathrm{arctg}(x))' = \frac{1}{1+x^2} \qquad (\mathrm{arccotg}(x))' = \frac{-1}{1+x^2}$$

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(iii) $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

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**Definice (derivace vyšších řádů):** n-tou derivaci funkce $f$ označujeme $f^{(n)}$ a definujeme:

$$f^{(n)} = \left[f^{(n-1)}\right]', \text{ kde } f^{(0)} = f.$$

# Mean value theorems

**Theorem (Rolle's mean value theorem):**
Let $f$ be continuous on $\langle a, b \rangle$ and differentiable on $(a, b)$, such that $f(a) = f(b)$. Then

$$\exists\, c \in (a, b) \ \text{ such that } \quad f'(c) = 0.$$

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**Theorem:**

$$f'(x) > 0 \text{ on interval } I \quad \Rightarrow \quad \text{f is increasing on } I$$

$$f'(x) < 0 \text{ on interval } I \quad \Rightarrow \quad \text{f is decreasing on } I$$

# l'Hospital's rule

**Theorem (l'Hospital's rule):** It holds

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

WHENEVER both of the following conditions are satisfied

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, or $\lim_{x\to a} |g(x)| = +\infty$

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- It holds also for one-sided limits and limits at improper points.
- It can be uses only for ratio and only for types $\frac{0}{0}$ or $\frac{\text{whatever}}{\infty}$.
- Ratio of derivatives is not the same as derivative of ratio!
- When the limit on the right hand side does not exist, the limit has to be calculated in a different way.

# Corollary of the l'Hospital rule

**Corollary:** Let $f$ be continuous on $\langle a, b)$, suppose $f'$ exists on $(a, b)$and that there exists $\lim_{x \to a+} f'(x)$. Then

$$f'_+(a) = \lim_{x \to a+} f'(x).$$

**Remark:** Analogously on $(a, b\rangle$ for $f'_-(b)$.