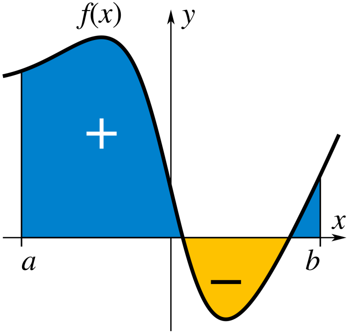
Definite integral.



wikipedia.org

Definite integral

Definition:
Let f be a continuous functionon interval $\langle a, b\rangle$, let F be its antiderivative. Then we define definite integral of f from a to b as a <u>number</u> $F(b) - F(a)$, we write

$$\int_a^b f(x)\,\mathrm{d}x = \big[F(x)\big]_a^b = F(b) - F(a)\,.$$

a - lower limit
b - upper limit

Important:
definite integral = number (increment of primitive function)
indefinite integral = function (primitive)

Remark:
The value of definite integral does not depend on the choice of primitive function.

Properties of definite integral

Theorem: It holds:

(i) $$\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$$

(ii) $$\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$$

(iii) $$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x, \quad c \in (a,b)$$

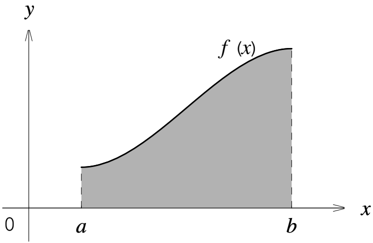
(iv) $$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

(v) If $f \geq 0$ on $\langle a,b\rangle$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$

Geometrical meaning

Thoerem: Let f be nonnegative continuous function on interval $\langle a, b\rangle$. Then the area of a planar figure bounded by graph of f, axis x and lines $x = a$, $x = b$ is equal

$$P = \int_a^b f(x)\,\mathrm{d}x\,.$$



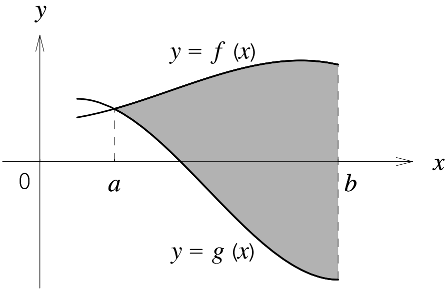
Remark: If f continuous negative, then

$$P = -\int_a^b f(x)\,\mathrm{d}x\,.$$

Geometrical meaning (2)

Theorem: Let f, g be continuous on interval $\langle a, b\rangle$ such that $\forall x \in \langle a, b\rangle : \; g(x) \leq f(x)$. Then the area of the depicted planar figure equals

$$P = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x\,.$$



Remark: The previous assertions are just special cases.

Theorem: Integration by parts

Suppose that u a v have on interval $\langle a, b\rangle$ continuous derivatives, then

$$\int_a^b u(x)v'(x)\,\mathrm{d}x = \big[u(x)v(x)\big]_a^b - \int_a^b u'(x)v(x)\,.$$

Theorem: Substitution

Let $f(t)$ be continuous on interval $\langle a, b\rangle$ and let $t = \varphi(x)$ have continuous derivative on interval $\langle \alpha, \beta\rangle$, and suppose that $\varphi(\langle \alpha, \beta\rangle) \subset \langle a, b\rangle$, then

$$\int_\alpha^\beta f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t$$

Improper integrals

Obsah nekonečných obrazců (např. plocha pod grafem $\frac{1}{x^2}$)

Definition: Let f be defined and continuous on an interval (a,b) (a can be $-\infty$, b can be ∞). Let F be its antiderivative on interval (a,b), then we define integral

$$\int_a^b f(x)\mathrm{d}x$$

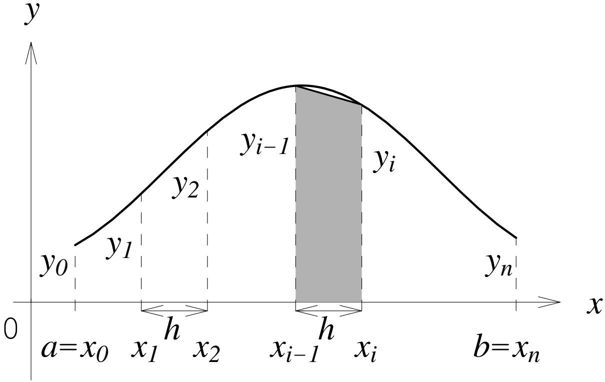
as a number

$$\left[F(x)\right]_a^b = \lim_{x\to b-} F(x) \; - \; \lim_{x\to a+} F(x),$$

whenever both limits exist and are finite.

In that case we say that the integral converges.
Otherwise (limit does not exist or is improper), we say that the integral diverges.

Numerical integration - trapezoidal method



Recall area of trapezoid:

$$A_{\text{tr}} = \frac{1}{2}h(b_1 + b_2)$$

Here, for the hatched area:

$$A_{\text{tr}} = \frac{1}{2}h(y_{i-1} + y_i)$$

Trapezoidal method

$$\int_a^b f(x)\mathrm{d}x \doteq A_{\mathrm{tr}_1} + A_{\mathrm{tr}_2} + \cdots + A_{\mathrm{tr}_\mathrm{n}} =$$
$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \cdots + \frac{h}{2}(y_{n-1} + y_n) =$$
$$= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

Theorem: (Chained trapezoidal rule)
Let f be continuous on $\langle a, b\rangle$. Divide the interval $\langle a, b\rangle$ into n equal parts by points

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$ (equidistant partition).

Denote $h = \frac{b-a}{n}$, so-called step, and $f(x_i) = y_i$ for $i = 0, 1, \ldots n$. Then

$$\int_a^b f(x)\mathrm{d}x \doteq \frac{h}{2}\big(y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\big).$$