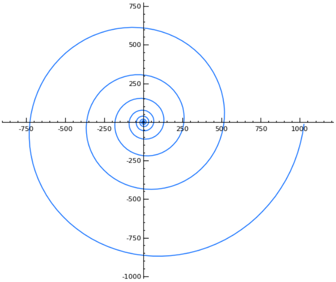
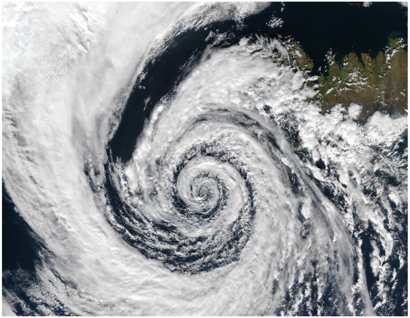
Parametric equations of planar curves





Logarithmic spiral, wikipedia.org

Mapping from $\mathbb{R}$ to $\mathbb{R}^2$

Definition: Let $I \subseteq \mathbb{R}$ be interval. Mapping that for every $t \in I$ uniquely assigns an ordered pair of numbers $\big(\varphi_1(t), \varphi_2(t)\big)$ is called a mapping of interval I to the plane $\mathbb{R}^2$. We denote

$$\varphi : I \to \mathbb{R}^2$$

$$\varphi : t \mapsto \big(\varphi_1(t), \varphi_2(t)\big) \quad = \big(x(t), y(t)\big)$$

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We refer to φ_1, φ_2 as coordinate functions, if they are continuous/differentiable, we say that φ is continuous/differentiable. We define the derivative φ' of map φ as

$$\varphi'(t) = \big(\varphi_1'(t), \varphi_2'(t)\big), \quad t \in I$$

Note: $\varphi'(t)$ is again map $\varphi' : I \to \mathbb{R}^2$.

Planar curve

Definition:
Let $\varphi(t) = \big(\varphi_1(t), \varphi_2(t)\big)$ be continuous mapping of interval I to $\mathbb{R}^2$, then the set

$$\mathcal{K} = \big\{\varphi(t) | t \in I\big\} = \big\{(x, y) \in \mathbb{R}^2 |\ x = \varphi_1(t), y = \varphi_2(t)\big\}.$$

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We refer to φ as a parametrization of $\mathcal{K}$ and the equations

$$\begin{aligned} x &= \varphi_1(t) \\ y &= \varphi_2(t), \quad t \in I \end{aligned}$$

are called parametric equations of $\mathcal{K}$.

Remark: Parametrization of a given curve is not unique.

Planar curves - kinematic interpretation

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Kinematic interpretation - **movement of a particle in plane dependent on time**, i.e. $(x, y) = (\varphi_1(t), \varphi_2(t))$ position of a particle at time t:

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Curve $\mathcal{K} \sim$ trajectory/path taken (set of points in plane)

Important examples of plane curves

- Line and its segments
  For $A \in p$ given point and $\vec{u}$ directional vector of p, then one of possible parameterizations is

$$p: \; X(t) = A + t \cdot \vec{u}, \; t \in \mathbb{R}$$

- Circle and its parts
  For k circle with center at point $C = [x_0, y_0]$ and radius r with equation $(x - x_0)^2 + (y - y_0)^2 = r^2$, then

$$\begin{aligned} C: \; x &= x_0 + r\cos t, \\ y &= y_0 + r\sin t, \; t \in \langle 0, 2\pi \rangle \end{aligned}$$

- Ellipse
  For e ellipse with center $C = [x_0, y_0]$ with semi-minor and semi-major axes a, b with equation $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$, then

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Important examples of plane curves

- If $\varphi_1(t)$, $t \in I$ is injective function $\Rightarrow$ $t = \varphi_1^{-1}(x)$, $x \in H(\varphi_1)$

  $\Rightarrow$ $y = \varphi_2(t) = \varphi_2(\varphi_1^{-1}(x))$, $x \in H(\varphi_1)$

  whence $\mathcal{K}$ is graph of a function $y = f(x)$.

- Similarly, if $\varphi_2(t)$, $t \in I$ is injective $\Rightarrow$

  $\Rightarrow$ $\mathcal{K}$ is graph of a function $x = f(y)$.

Tangent vector

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$$\vec{v}(t_0) = \varphi'(t_0) = (\varphi_1'(t_0), \varphi_2'(t_0))$$

is the tangent vector to the curve $\mathcal{K}$ at $t = t_0$.

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  $\vec{v}(t_0) = \varphi'(t_0)$ is tangent vector to the curve $\mathcal{K}$ at point $\varphi(t_0)$;

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