Continuity and limits of functions

Neighbourhoods

Notation: $a \in \mathbb{R},\ \varepsilon > 0$

- $\mathcal{O}_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$ ε-neighbourhood of the point a

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- $x \to a$ x tends to a
  id est x takes values arbitrarily close to a
  Similarly: $x \to a+,\ x \to a-,\ x \to +\infty,\ x \to -\infty$

Continuity at a point

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Definition: Let f be function defined in a neighbourhood $\mathcal{O}(a)$ of the point a. We say that f is continuous at the point $a \in D(f)$, iff

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \text{ such that } \; f\big(\mathcal{O}_\delta(a)\big) \subseteq \mathcal{O}_\varepsilon\big(f(a)\big)$$

Roughly: At points "close to" a has f values "close to" $f(a)$.

Limit of a function

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Definition:
Let $a \in \mathbb{R}$ and let f be defined on some punctured neighbourhood $\mathcal{P}(a) \subseteq D(f)$. We say that function f has the limit $L \in \mathbb{R}$ at point a if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } f\big(\mathcal{P}_\delta(a)\big) \subset \mathcal{O}_\varepsilon(L).$$

We write

$$\lim_{x \to a} f(x) = L.$$

Roughly: At points "close to" a, function f has values "close to" L.

$$x \to a \quad \Rightarrow \quad f(x) \to L$$

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Theorem:
Function f is continuous at point a if and only if $\lim_{x \to a} f(x) = f(a)$.

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I. If $a, L \in \mathbb{R}$, we say proper limit at proper point

$$\text{e.g. } \lim_{x \to \pi} \cos(x) = -1$$

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II. If $a \in \mathbb{R}$, $L = \pm\infty$ ... improper limit at proper point

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III. If $a = \pm\infty$, $L \in \mathbb{R}$ ... proper limit at improper point

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IV. If $a, L = \pm\infty$ ... improper limit at improper point

$$\text{e.g. } \lim_{x \to \infty} \mathrm{e}^x = \infty$$

Improper limits - case II

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Definice: Let f be defined on $\mathcal{P}(a)$ then

(i) $\lim_{x \to a} f(x) = \infty$, if

$$\forall\, K > 0 \;\; \exists \delta > 0 \quad \text{such that} \;\; \forall\, x \in \mathcal{P}_\delta(a) \;\; \text{is} \;\; f(x) > K$$

(ii) $\lim_{x \to a} f(x) = -\infty$, if

$$\forall\, L < 0 \;\; \exists \delta > 0 \quad \text{such that} \;\; \forall\, x \in \mathcal{P}_\delta(a) \;\; \text{is} \;\; f(x) < L$$

Proper limit at improper point - case III

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Definition:

$$\lim_{x \to \infty} f(x) = L_1$$

if

$$\forall\, \varepsilon\, \exists\, x_1 > 0 \text{ such that } \forall\, x > x_1 \text{ holds } f(x) \in \mathcal{O}_\varepsilon(L_1)$$

Similarly:

$$\lim_{x \to -\infty} f(x) = L_2$$

if

$$\forall\, \varepsilon\, \exists\, x_2 < 0 \text{ such that } \forall\, x < x_2 \text{ holds } f(x) \in \mathcal{O}_\varepsilon(L_2)$$

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Definition:

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Calculating the limits

Theorem: Let $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, $\quad (a, A, B \in \mathbb{R} \cup \pm\infty)$.
Then:

- $\lim_{x\to a} \left(f(x) \pm g(x)\right) = A \pm B$,
- $\lim_{x\to a} \left(f(x) \cdot g(x)\right) = A \cdot B$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$

if the right hand side has sense - see "arithmetics of infinity" and "dividing by zero".

Calculating the limits

The following rules are abbreviations for the assertions in the sense of previous theorem

"Arithmetics of infinity": $C \in \mathbb{R}$, $C > 0$.

$$\begin{array}{rcccr} \infty & + & C & = & \infty \\ \infty & - & C & = & \infty \\ \infty & + & \infty & = & \infty \\ -\infty & - & \infty & = & -\infty \\ -\infty & + & C & = & -\infty \\ -\infty & - & C & = & -\infty \end{array}$$

$$\begin{array}{rcccr} \infty & \cdot & C & = & \infty \\ \infty & \cdot & (-C) & = & -\infty \\ -\infty & \cdot & C & = & -\infty \\ -\infty & \cdot & (-C) & = & \infty \\ \infty & \cdot & \infty & = & \infty \\ \infty & \cdot & (-\infty) & = & -\infty \\ (-\infty) & \cdot & (-\infty) & = & \infty \end{array}$$

$$\infty - \infty = ???$$

$$\infty \cdot 0 = ???$$

$$\frac{C}{\pm\infty} = \frac{-C}{\pm\infty} = \frac{0}{\pm\infty} = 0$$

$$\frac{\infty}{\infty} = ???$$

Calculating the limits

"Dividing by zero": $C \in \mathbb{R}$, $C > 0$.

$$\begin{aligned} \frac{C}{0+} &= \infty & \qquad \frac{\infty}{0+} &= \infty \\ \frac{-C}{0+} &= -\infty & \qquad \frac{\infty}{0-} &= -\infty \\ \frac{C}{0-} &= -\infty & \qquad \frac{-\infty}{0+} &= -\infty \\ \frac{-C}{0-} &= \infty & \qquad \frac{-\infty}{0-} &= \infty \\ \frac{0}{0} &= ??? \end{aligned}$$

Indeterminate forms: $\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty$

...will be calculated by simplifying the expression or by the l'Hospital rule

Limit of sequence

Definition:

$$\lim_{n \to \infty} a_n = L \in \mathbb{R} \text{ (proper limit), if}$$

$$\forall\, \varepsilon\ \exists\, n_0 \in \mathbb{N} \text{ such that } \forall\, n > n_0 \text{ it holds} a_n \in \mathcal{O}_\varepsilon(L)\,.$$

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$L \in \mathbb{R}$ ... convergent sequence

$L = \pm\infty$ or the limit does not exist ... divergent sequence

Euler's number

It can be proven that the following limit exists and it is finite

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Definition: Denote

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Number $\mathrm{e} \doteq 2,71828$ is called Euler's number.