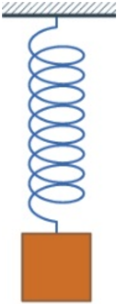
Applications of differential equations



Applications of differential equations

Often in applications, the independent variable has the meaning of time, then the time derivative has the meaning of instanteneous rate of change of the corresponding quantity.

Important: If $y(t)$ is possition at time t, then $y'(t)$ is the instanteneous velocity and $y''(t)$ instanteneous accelaration.

Uniformly accelarated motion

For given accelaration $a > 0$ consider the following problem with given initial position y_0 and given initial velocity v_0:

$$y''(t) = a, \qquad y(0) = y_0, \qquad y'(0) = v_0.$$

Direct integration yields

$$\begin{aligned} y''(t) = a, \quad \Rightarrow \quad y'(t) &= at + C_1, \\ y(t) &= \frac{1}{2}at^2 + C_1 t + C_2, \end{aligned}$$

and from the initial conditions we deduce $C_1 = v_0,\ C_2 = y_0$.

$$\boxed{y(t) = \frac{1}{2}at^2 + v_0 t + y_0} \text{ (viz SŠ)}$$

Remark: It is a 2^{nd} order linear equation with constant coefficients. How the characteristic equation looks like? The method of undetermined coefficients can be applied.

Harmonic oscillator

Denote: $y = y(t)$ - deflection at time t,
m - mass of the body, k - stiffness of the spring

Newton's law $\boxed{F = ma = my''(t)}$ Hooke's law $\boxed{F_H = -ky}$

Comparing the forces - the motion described by equation

$$my'' + ky = 0$$

$$y'' + \frac{k}{m}y = 0$$

$$\omega := \sqrt{\frac{k}{m}} \quad \Rightarrow \quad y'' + \omega^2 y = 0$$

$$\lambda^2 + \omega^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm\omega i$$

General solution has the form

$$y_{OH} = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

Resonance

Let us add a driving $\frac{2\pi}{p}$-periodic force with $p > 0$

$$y'' + \omega^2 y = \sin(pt).$$

The method of undetermined coefficients can be used for the particular solution of the non-homogeneous equation in the form

$$y_{PN} = t^k \big(A\cos(pt) + B\sin(pt)\big),$$

where k is the multiplicity of number pi as the root of characteristic equation ($\lambda^2 + \omega^2 = 0$).

In dependence on values of p and ω we consider two cases

- $p \neq \omega$
- $p = \omega$

Case $p \neq \omega$

For $p \neq \omega$ is the number pi not a root of characteristic equation, $k = 0$ and we get

$$y_{PN} = \frac{1}{\omega^2 - p^2} \sin(pt),$$

$$\boxed{y_{ON} = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{1}{\omega^2 - p^2} \sin(pt)}$$

The solution is a bounded function, for any initial conditions.

Remark: For ω close to p the maximal amplitude of the solution grows ($\omega^2 - p^2$ in denominator).

Case $p = \omega$ - resonance

In this case, the number $pi = \omega i$ is single root, $k = 1$ and one can get

$$y_{PN} = -\frac{1}{2\omega} t \cos(\omega t)$$

and

$$\boxed{y_{ON} = C_1 \cos(\omega t) + C_2 \sin(\omega t) - \frac{1}{2\omega} t \cos(\omega t)}$$

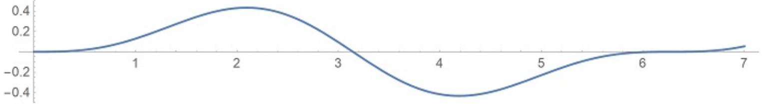
The solutionis for any initial condition unbounded, *resonance* occurs.

Remark: The unboundedness of the solution is independent of (positive) amplitude of the forcing, what matters is the co-ordination of the periodicity of the forcing and the period of the oscilator itself.

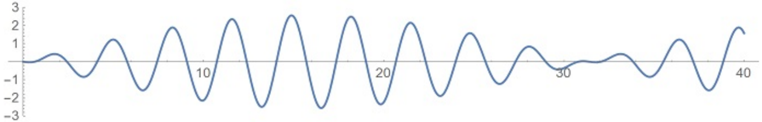
Illustration

Consider initial conditions $y(0) = y'(0) = 0$, $\omega = 2$ a four different periods of the forcing.

- $p = 1$



- $p = 1.8$

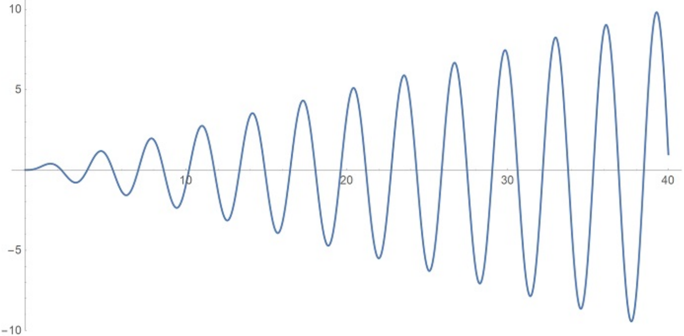


When p approaches ω, the maximal amplitude grows.

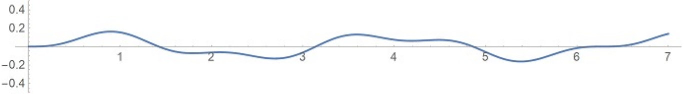
Illustration

Consider initial conditions $y(0) = y'(0) = 0$, $\omega = 2$ and four different periods of the forcing...

- $p = 2$



- $p = 5$



Damped oscilator

Consider damping proportional to the velocity: $F_d = -cv$, with $c > 0$ damping coefficient

$$\begin{aligned} ma &= F_H + F_t, \\ my'' &= -ky - cy', \\ my'' + cy' + ky &= 0, \\ y'' + \frac{c}{m}y' + \frac{k}{m}y &= 0. \end{aligned}$$

Denote: $\omega = \sqrt{\frac{k}{m}}$ ... period of oscillations without damping

$\zeta = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega}$ ... damping ratio, then

$$y'' + 2\zeta\omega y' + \omega^2 y = 0$$

Damped oscilator

$$y'' + 2\zeta\omega y' + \omega^2 y = 0$$

It is a second order linear homogeneous equation with constant coefficients.

Characteristic equation $\lambda^2 + 2\zeta\omega\lambda + \omega^2\lambda = 0$ has discriminant equal

$$D = (2\zeta\omega)^2 - 4\omega^2 = 4\omega^2(\zeta^2 - 1).$$

Let us consider three cases:

- $0 < \zeta < 1$
- $\zeta = 1$
- $\zeta > 1$

Case $\zeta < 1$ - *underdamped*

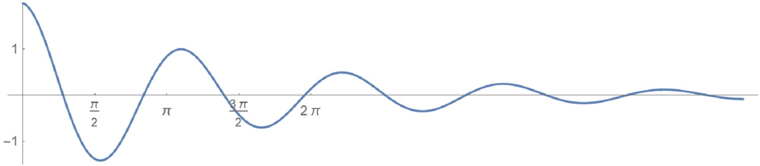
$D < 0$, two complex roots

$$\lambda_{1,2} = \frac{-2\zeta\omega \pm \sqrt{4\omega^2(\zeta^2 - 1)}}{2} = -\zeta\omega \pm \omega\sqrt{1 - \zeta^2}\, i$$

and general solution in the form

$$y(t) = \mathrm{e}^{-\zeta\omega t}\left(C_1 \cos\left(\omega\sqrt{1-\zeta^2}\, t\right) + C_2 \sin\left(\omega\sqrt{1-\zeta^2}\, t\right)\right)$$

We can see the situation for $\zeta = \frac{1}{5}$, $\omega = 2$ a $C_1 = 2$, $C_2 = 0$, on the figure.



Case $\zeta = 1$ - *critical damping*

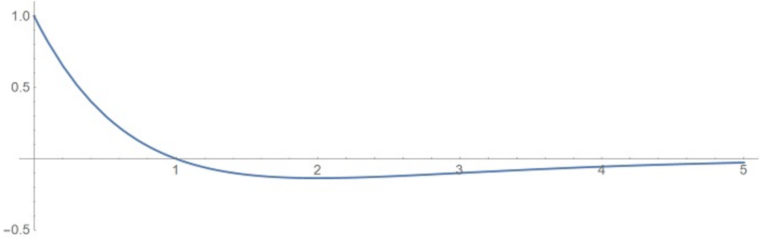
Characteristic equation has the form $\lambda^2 + 2\omega\lambda + \omega^2\lambda = 0$ thus we get one double root $\lambda = -\omega$.
And general solution reads

$$y(t) = C_1 \mathrm{e}^{-\omega t} + C_2 t \mathrm{e}^{-\omega t},$$

in particular for initial conditions $y(0) = y_0,\ y'(0) = v_0$

$$y(t) = \big(y_0(1 + \omega t) + v_0\big)\mathrm{e}^{-\omega t}.$$

The situation for $\omega = 1$ a $y_0 = 1,\ v_0 = -2$, is on the picture.



Case $\zeta > 1$ - *overdamped*

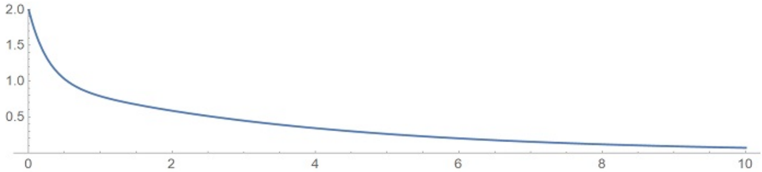
Two distinct roots (both negative)

$$\lambda_{1,2} = \frac{-2\zeta\omega \pm \sqrt{4\omega^2(\zeta^2 - 1)}}{2} = \omega\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

and general solution

$$y(t) = C_1 \mathrm{e}^{\lambda_1 t} + C_2 \mathrm{e}^{\lambda_2 t},$$

On the figure, $\omega = 1$, $\zeta = 2$ and $C_1 = 1,\ C_2 = 1$ is depicted.



Chemical kinetics

Consider simplest chemical reaction of order n driven by rate equation

$$A'(t) = -k\big(A(t)\big)^n,$$

where $A(t) \dots$ concentration of the reactant at time t,
$k > 0 \dots$ reaction rate constant

We will consider three cases $n \in \{0, 1, 2\}$, always with initial condition $A(0) = A_0 > 0$.

Remark: In all cases, it is a separable differential equation of the first order.

Case $n = 0$

Problem

$$\begin{aligned} A' &= -k, \\ A(0) &= A_0 \end{aligned}$$

has unique solution

$$A(t) = A_0 - kt,$$

formula has good sense for $t \leq \frac{A_0}{k}$, where is the solution non-negative.

In the remaining cases, the equation has one constant solution $A \equiv 0$, thus consider only positive solutions.

Case $n = 1$

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -kA \\ \int \frac{1}{A}\mathrm{d}A &= -k\int \mathrm{d}t \\ \ln(A) &= -kt + \ln(A_0) \\ A(t) &= A_0\mathrm{e}^{-kt},\ t \in \mathbb{R} \end{aligned}$$

Remark: Equation is in this case also linear homogeneous, even with constant coefficients,it can be solved using the characteristic equation $\lambda + k = 0$.

Case $n = 2$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -kA^2$$

$$-\int \frac{1}{A^2}\mathrm{d}A = k\int \mathrm{d}t$$

$$\frac{1}{A} = kt + \frac{1}{A_0}$$

$$A(t) = \frac{A_0}{1 + A_0 kt}$$

Solution has sense for all positive times.

Illustration

On the picture, the situation for $A_0 = 2$ and $k = 1$ is depicted for chemical reaction of order zero, one and two.