12. série - poznámky k řešení

Příklad 1: Ověřte, že funkce $U(x,y) = 3y\sqrt[3]{x} + 2x\sqrt{y^2+1}$ je v oblasti $G = \{(x,y) \in \mathbb{R}^2,\ x > 0\}$ potenciálem vektorového pole

$$\vec{F}(x,y) = \left(\frac{2\sqrt{y^2+1}\sqrt[3]{x^2} + y}{\sqrt[3]{x^2}},\ \frac{2xy + 3\sqrt{y^2+1}\sqrt[3]{x}}{\sqrt{y^2+1}} \right).$$

Pomocí potenciálu U spočtěte integrál $\int_{\mathcal{K}} \vec{F} \cdot \mathrm{d}\vec{r}$, kde $\mathcal{K}$ je oblouk $y = x^2 - 4x + 3$, $x \in \langle 1, 3 \rangle$ orientovaný souhlasně s rostoucím x.

Řešení: $G \subset \mathcal{D}_{\vec{F}} = \{(x,y) \in \mathbb{R}^2,\ x \neq 0\}$ Máme ověřit, že $\operatorname{grad} U = \vec{F}$.

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}\left(3y\sqrt[3]{x} + 2x\sqrt{y^2+1}\right) = \frac{y}{\sqrt[3]{x^2}} + 2\sqrt{y^2+1} = \frac{y + 2\sqrt{y^2+1}\sqrt[3]{x^2}}{\sqrt[3]{x^2}}$$

$$\frac{\partial U}{\partial x} = F_1, \qquad \forall (x,y) \in G$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(3y\sqrt[3]{x} + 2x\sqrt{y^2+1}\right) = 3\sqrt[3]{x} + \frac{4xy}{2\sqrt{y^2+1}} = \frac{3\sqrt{y^2+1}\sqrt[3]{x} + 2xy}{\sqrt{y^2+1}}$$

$$\frac{\partial U}{\partial y} = F_2, \qquad \forall (x,y) \in G$$

Funkce U je tedy potenciálem vektorového pole $\vec{F}$ na oblasti G. Spočtěme počáteční a koncový bod křivky $P.B. = (1, 1-4+3) = (1,0)$,
$K.B. = (3, 9-12+3) = (3,0)$. Křivka $\mathcal{K}$ leží v oblasti G ($\mathcal{K} \subset G$), proto

$$\int_{\mathcal{K}} \vec{F}\mathrm{d}r = U(K.B.) - U(P.B.) = U(3,0) - U(1,0) = (0+6) - (0+2) = 4.$$

Příklad 2: Ověřte, že je dané pole $\vec{F}$ potenciální a určete jeho potenciál U tak, aby hodnota potenciálu v počátku byla 1.

Řešení:

a) $\vec{F}(x,y) = \left(\frac{2x(1-\mathrm{e}^y)}{(1+x^2)^2},\ \frac{\mathrm{e}^y}{1+x^2}\right)$, $\mathcal{D}_{\vec{F}} = \mathbb{R}^2$ - **konvexní oblast**

$$\frac{\partial F_1}{\partial y} = \frac{-2x\mathrm{e}^y}{(1+x^2)^2} = \frac{\partial F_2}{\partial x}, \qquad \forall (x,y) \in \mathbb{R}^2$$

Vektorové pole $\vec{F}$ je potenciální na $\mathbb{R}^2$.

I. metoda - z definice

$$\begin{aligned}
\frac{\partial U}{\partial x} &= F_1 \\
U(x,y) &= \int F_1 \mathrm{d}x = \int \frac{2x(1-\mathrm{e}^y)}{(1+x^2)^2}\mathrm{d}x = -\frac{1-\mathrm{e}^y}{1+x^2} + \varphi(y) \\
\frac{\partial U}{\partial y} &= F_2 \\
\frac{\mathrm{e}^y}{1+x^2} + \varphi'(y) &= \frac{\mathrm{e}^y}{1+x^2} \\
\varphi'(y) &= 0 \quad \Rightarrow \quad \varphi(y) = C \\
U(x,y) &= \frac{\mathrm{e}^y - 1}{1+x^2} + C, \qquad U(0,0) = 0 + C = 1 \\
U(x,y) &= \frac{\mathrm{e}^y - 1}{1+x^2} + 1, \qquad (x,y) \in \mathbb{R}^2
\end{aligned}$$

II. metoda - nezávislost křivkového integrálu na cestě

$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$

$$\mathcal{K}_1 : x = t,\quad y = 0, \quad t \in \langle 0, \tilde{x} \rangle \qquad\qquad \mathcal{K}_2 : x = \tilde{x},\quad y = t, \quad t \in \langle 0, \tilde{y} \rangle$$

$$\begin{aligned}
\int_{\mathcal{K}} \vec{F} \cdot \mathrm{d}\vec{r} &= U(\tilde{x},\tilde{y}) - U(0,0) \\
U(\tilde{x},\tilde{y}) &= \int_{\mathcal{K}} \vec{F}\mathrm{d}\vec{r} + U(0,0) = \int_{\mathcal{K}_1} \vec{F}\mathrm{d}\vec{r} + \int_{\mathcal{K}_2} \vec{F}\mathrm{d}\vec{r} + 1 \\
\int_{\mathcal{K}_1} \vec{F}\mathrm{d}\vec{r} &= \int_0^{\tilde{x}} \frac{2t(1-\mathrm{e}^0)}{(1+t^2)^2}\mathrm{d}t = 0 \\
\int_{\mathcal{K}_2} \vec{F}\mathrm{d}\vec{r} &= \int_0^{\tilde{y}} \frac{\mathrm{e}^t}{1+\tilde{x}^2}\mathrm{d}t = \left[\frac{\mathrm{e}^t}{1+\tilde{x}^2}\right]_{t=0}^{\tilde{y}} = \frac{\mathrm{e}^{\tilde{y}} - 1}{1+\tilde{x}^2} \\
U(x,y) &= \frac{\mathrm{e}^y - 1}{1+x^2} + 1, \qquad (x,y) \in \mathbb{R}^2
\end{aligned}$$

b) $\vec{F}(x,y,z) = (2xy, x^2, -\frac{1}{1+z^2})$. $\mathcal{D}_{\vec{F}} = \mathbb{R}^3$ - **konvexní oblast**

$$\frac{\partial F_1}{\partial y} = 2x = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = 0 = \frac{\partial F_3}{\partial y}, \qquad \forall (x,y,z) \in \mathbb{R}^3$$

Vektorové pole $\vec{F}$ je potenciální na $\mathbb{R}^3$.

I. metoda - z definice

$$\frac{\partial U}{\partial x} = F_1$$

$$U(x,y,z) = \int F_1 \mathrm{d}x = \int 2xy\mathrm{d}x = x^2y + \varphi(y,z)$$

$$\frac{\partial U}{\partial y} = F_2$$

$$x^2 + \frac{\partial \varphi}{\partial y}(y,z) = x^2$$

$$\frac{\partial \varphi}{\partial y}(y,z) = 0 \quad \Rightarrow \quad \varphi(y,z) = \psi(z)$$

$$\frac{\partial U}{\partial z} = F_3$$

$$\psi'(z) = -\frac{1}{1+z^2}$$

$$\psi(z) = -\int \frac{1}{1+z^2}\mathrm{d}z = \operatorname{arccotg} z + C$$

$$U(x,y,z) = x^2y + \operatorname{arccotg} z + C, \qquad U(0,0,0) = \frac{\pi}{2} + C = 1$$

$$U(x,y,z) = x^2y + \operatorname{arccotg} z - \frac{\pi}{2} + 1, \qquad (x,y,z) \in \mathbb{R}^3$$

II. metoda - nezávislost křivkového integrálu na cestě

$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3$

$$\mathcal{K}_1 : x = t,\ y = 0,\ z = 0,\quad t \in \langle 0, \tilde{x} \rangle \qquad \mathcal{K}_2 : x = \tilde{x},\ y = t,\ z = 0,\quad t \in \langle 0, \tilde{y} \rangle \qquad \mathcal{K}_3 : x = \tilde{x},\ y = \tilde{y},\ z = t,\quad t \in \langle 0, \tilde{z} \rangle$$

$$\int_{\mathcal{K}} \vec{F} \cdot \mathrm{d}\vec{r} = U(\tilde{x},\tilde{y},\tilde{z}) - U(0,0,0) \quad \Rightarrow \quad U(\tilde{x},\tilde{y},\tilde{z}) = \int_{\mathcal{K}_1} \vec{F}\mathrm{d}\vec{r} + \int_{\mathcal{K}_2} \vec{F}\mathrm{d}\vec{r} + \int_{\mathcal{K}_3} \vec{F}\mathrm{d}\vec{r} + 1$$

$$\int_{\mathcal{K}_1} \vec{F}\mathrm{d}\vec{r} = \int_0^{\tilde{x}} 0\mathrm{d}t = 0, \qquad \int_{\mathcal{K}_2} \vec{F}\mathrm{d}\vec{r} = \int_0^{\tilde{y}} \tilde{x}^2\mathrm{d}t = \tilde{x}^2\tilde{y}$$

$$\int_{\mathcal{K}_3} \vec{F}\mathrm{d}\vec{r} = \int_0^{\tilde{z}} -\frac{1}{1+t^2}\mathrm{d}t = \Big[-\operatorname{arctg} t\Big]_{t=0}^{\tilde{z}} = -\operatorname{arctg} \tilde{z}$$

$$U(x,y,z) = x^2y - \operatorname{arctg} z + 1, \qquad (x,y,z) \in \mathbb{R}^3$$

c) $\vec{F}(x,y,z) = (y^2 + 2xz^2,\ 2xy - 1,\ 2x^2z + z^3)\,.\mathcal{D}_{\vec{F}} = \mathbb{R}^3$ - **konvexní oblast**

$$\frac{\partial F_1}{\partial y} = 2y = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = 4xz = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = 0 = \frac{\partial F_3}{\partial y}, \qquad \forall (x,y,z) \in \mathbb{R}^3$$

Vektorové pole $\vec{F}$ je potenciální na $\mathbb{R}^3$.

I. metoda - z definice

$$\begin{aligned}
\frac{\partial U}{\partial x} &= F_1 \\
U(x,y,z) &= \int F_1 \mathrm{d}x = \int y^2 + 2xz^2 \mathrm{d}x = xy^2 + x^2z^2 + \varphi(y,z) \\
\frac{\partial U}{\partial y} &= F_2 \\
2xy + \frac{\partial \varphi}{\partial y}(y,z) &= 2xy - 1 \\
\frac{\partial \varphi}{\partial y}(y,z) &= -1 \quad \Rightarrow \quad \varphi(y,z) = \int -1 \mathrm{d}y = -y + \psi(z) \\
U(x,y,z) &= xy^2 + x^2z^2 - y + \psi(z) \\
\frac{\partial U}{\partial z} &= F_3 \quad \Rightarrow \quad 2x^2z + \psi'(z) = 2x^2z + z^3 \\
\psi(z) &= \int z^3 \mathrm{d}z = \frac{z^4}{4} + C \\
U(x,y,z) &= xy^2 + x^2z^2 - y + \frac{z^4}{4} + C, \qquad U(0,0,0) = C = 1 \\
U(x,y,z) &= xy^2 + x^2z^2 - y + \frac{z^4}{4} + 1, \qquad (x,y,z) \in \mathbb{R}^3
\end{aligned}$$

II. metoda - nezávislost křivkového integrálu na cestě

$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3$

$$\begin{aligned}
\mathcal{K}_1 : x &= t, & \mathcal{K}_2 : x &= \tilde{x}, & \mathcal{K}_3 : x &= \tilde{x}, \\
y &= 0, & y &= t & y &= \tilde{y}, \\
z &= 0, \quad t \in \langle 0, \tilde{x} \rangle & z &= 0, \quad t \in \langle 0, \tilde{y} \rangle & z &= t, \quad t \in \langle 0, \tilde{z} \rangle
\end{aligned}$$

$$\int_{\mathcal{K}} \vec{F} \cdot \mathrm{d}\vec{r} = U(\tilde{x},\tilde{y},\tilde{z}) - U(0,0,0) \quad \Rightarrow \quad U(\tilde{x},\tilde{y},\tilde{z}) = \int_{\mathcal{K}_1} \vec{F}\mathrm{d}\vec{r} + \int_{\mathcal{K}_2} \vec{F}\mathrm{d}\vec{r} + \int_{\mathcal{K}_3} \vec{F}\mathrm{d}\vec{r} + 1$$

$$\int_{\mathcal{K}_1} \vec{F}\mathrm{d}\vec{r} = \int_0^{\tilde{x}} 0 \mathrm{d}t = 0, \qquad \int_{\mathcal{K}_2} \vec{F}\mathrm{d}\vec{r} = \int_0^{\tilde{y}} 2\tilde{x}t - 1 \mathrm{d}t = \Big[\tilde{x}t^2 - t\Big]_{t=0}^{\tilde{y}} = \tilde{x}\tilde{y}^2 - \tilde{y}$$

$$\int_{\mathcal{K}_3} \vec{F}\mathrm{d}\vec{r} = \int_0^{\tilde{z}} 2\tilde{x}^2 t + t^3 \mathrm{d}t = \Big[\tilde{x}^2t^2 + \frac{t^4}{4}\Big]_{t=0}^{\tilde{z}} = \tilde{x}^2\tilde{z}^2 + \frac{\tilde{z}^4}{4}$$

$$U(x,y,z) = xy^2 - y + x^2z^2 + \frac{z^4}{4} + 1, \qquad (x,y,z) \in \mathbb{R}^3$$

Příklad 3: $\vec{F}(x,y,z) = [-y, (\frac{1}{y}-x), 2z]$. Má $\vec{F}$ potenciál na oblasti $\{(x,y,z) \in \mathbb{R}^3, y > 0\}$?
Určete $\displaystyle\int_{\mathcal{C}} -y\mathrm{d}x + (\frac{1}{y} - x)\mathrm{d}y + 2z\mathrm{d}z$, pro $\mathcal{C}$ danou parametrickými rovnicemi

$$x = 1 + \cos t, \qquad y = 2 + \cos t, \qquad z = 5\sin t, \qquad t \in \langle 0, 4\pi \rangle.$$

Řešení:
$\mathcal{D}_{\vec{F}} = \{(x,y,z) \in R^3, y \neq 0\}$
$G = \{(x,y,z) \in \mathbb{R}^3, y > 0\} \subset \mathcal{D}_{\vec{F}}$, G je **konvexní oblast.**

$$\frac{\partial F_1}{\partial y} = -1 = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = 0 = \frac{\partial F_3}{\partial y}, \qquad \forall (x,y,z) \in G$$

Pole $\vec{F}$ **je potenciální pole na** G, $K.B. = P.B. = [2,3,0]$. **Křivka** $\mathcal{C}$ **je uzavřená a** $\mathcal{C} \subset G$**, proto**

$$\int_{\mathcal{C}} \vec{F}\mathrm{d}\vec{r} = 0.$$

Příklad 4: Pole $\vec{F}(x,y,z) = [x - z, 1 - xy, y]$ není potenciální

$$\frac{\partial F_1}{\partial y} = 0, \quad \text{ale} \quad \frac{\partial F_2}{\partial x} = -y.$$

$$\int_{\mathcal{K}} \vec{F}\mathrm{d}\vec{r} = \int_0^1 (t - t^3) + (1 - t^3)2t + t^2 \cdot 3t^2\mathrm{d}t = \int_0^1 (t - t^3) + 2t - 2t^4 + 3t^4\mathrm{d}t =$$

$$= \int_0^1 3t - t^3 + t^4\mathrm{d}t = \left[\frac{3t^2}{2} - \frac{t^4}{4} + \frac{t^5}{5}\right]_{t=0}^1 = \frac{3}{2} - \frac{1}{4} + \frac{1}{5} = \frac{30 - 5 + 4}{20} = \frac{29}{20}.$$