10. série

2. příklad

$$x^4+y^4+z^4+x-y-z=0, (x_0,y_0,z_0)=(-1,1,1), \quad z=f(x,y), \quad \frac{\partial^2 f}{\partial x \partial y}(-1,1)=?$$

Označme

$$F(x,y)=x^4+y^4+z^4+x-y-z.$$

- $F \in C^2(\mathbb{R}^3)$ ✓
- $F(-1,1,1)=1+1+1-1-1-1=0$ ✓
- $\frac{\partial F}{\partial z}(x,y,z)=4z^3-1 \quad \Rightarrow \quad \frac{\partial F}{\partial z}(-1,1,1)=3 \neq 0$ ✓

Podle Věty o implicitní funkci tedy rovnice definuje na okolí bodu $(-1,1,1)$ **spojitou funkci** $z=f(x,y)$.

I. metoda derivování

$$\begin{aligned}
\frac{\partial F}{\partial x}(x,y,z)&=4x^3+1, \quad \Rightarrow \quad \frac{\partial F}{\partial x}(-1,1,1)=-4+1=-3\\
\frac{\partial F}{\partial y}(x,y,z)&=4y^3-1, \quad \Rightarrow \quad \frac{\partial F}{\partial y}(-1,1,1)=4-1=3\\
\frac{\partial F}{\partial z}(x,y,z)&=4z^3-1, \quad \Rightarrow \quad \frac{\partial F}{\partial z}(-1,1,1)=4-1=3
\end{aligned}$$

$$\frac{\partial f}{\partial x}(x,y)=-\frac{\frac{\partial F}{\partial x}\big(x,y,f(x,y)\big)}{\frac{\partial F}{\partial z}\big(x,y,f(x,y)\big)}=-\frac{4x^3+1}{4\big(f(x,y)\big)^3-1}, \quad \Rightarrow \quad \frac{\partial f}{\partial x}(-1,1)=-\frac{-3}{3}=1$$

$$\frac{\partial f}{\partial y}(x,y)=-\frac{\frac{\partial F}{\partial y}\big(x,y,f(x,y)\big)}{\frac{\partial F}{\partial z}\big(x,y,f(x,y)\big)}=-\frac{4y^3-1}{4\big(f(x,y)\big)^3-1}, \quad \Rightarrow \quad \frac{\partial f}{\partial y}(-1,1)=-\frac{3}{3}=-1$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y)=\frac{\partial}{\partial y}\Big(-\frac{4x^3+1}{4\big(f(x,y)\big)^3-1}\Big)=-\frac{-(4x^3+1)12\big(f(x,y)\big)^2\frac{\partial f}{\partial y}(x,y)}{\Big(4\big(f(x,y)\big)^3-1\Big)^2}$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x \partial y}(-1,1)=\frac{(-3)\cdot 12 \cdot (-1)}{3^2}=4$$

II. metoda derivování (značíme $z'_x = \dfrac{\partial f}{\partial x}$, $A = (-1,1)$ atp.)

$$x^4 + y^4 + z^4 + x - y - z = 0$$

$$4x^3 + 4z^3 z'_x + 1 - z'_x = 0 \quad \Rightarrow \quad -4 + 4z'_x(A) + 1 - z'_x(A) = 0$$

$$+\,3z'_x(A) - 3 = 0, \qquad \frac{\partial f}{\partial x}(-1,1) = 1$$

$$4y^3 + 4z^3 z'_y - 1 - z'_y = 0 \quad \Rightarrow \quad 4 + 4z'_y(A) - 1 - z'_y(A) = 0$$

$$+\,3z'_y(A) + 3 = 0, \qquad \frac{\partial f}{\partial y}(-1,1) = -1$$

$$12z^2 z'_y z'_x + 4z^3 z''_{xy} - z''_{xy} = 0 \quad \Rightarrow \quad -12 + 4z''xy(A) - z''_{xy}(A) = 0$$

$$-\,12 + 3z''_{xy}(A) = 0, \qquad \frac{\partial^2 f}{\partial x \partial y}(-1,1) = 4$$

3. příklad

$$u = w + \operatorname{arctg} \frac{v}{w+u}, \ (u_0, v_0, w_0) = (1,0,1), \quad u = g(v,w), \quad g(0.01, 0.98) \doteq ?$$

Označme $F(u,v,w) = u - w - \operatorname{arctg} \frac{v}{w+u}$

- $\mathcal{D}_F = \{(u,v,w) \in \mathbb{R}^3, w \neq -u\}$, $F \in C^1(\mathcal{D}_F)$ ✓
- $F(1,0,1) = 1 - 1 - 0 = 0$ ✓
- $\dfrac{\partial F}{\partial u}(u,v,w) = 1 - \dfrac{1}{1 + \left(\frac{v}{w+u}\right)^2} \dfrac{-v}{(w+u)^2} \quad \Rightarrow \quad \dfrac{\partial F}{\partial u}(1,0,1) = 1 \neq 0$ ✓

Podle Věty o implicitní funkci tedy rovnice definuje na okolí bodu $(1,0,1)$ **spojitou funkci** $u = g(v,w)$.

Pro derivace vytvořující funkce $F(u,v,w) = u - w - \operatorname{arctg} \frac{v}{w+u}$ máme

$$\frac{\partial F}{\partial u}(u,v,w) = 1 - \frac{1}{1 + \left(\frac{v}{w+u}\right)^2} \frac{-v}{(w+u)^2}, \qquad \Rightarrow \quad \frac{\partial F}{\partial u}(1,0,1) = 1$$

$$\frac{\partial F}{\partial v}(u,v,w) = -\frac{1}{1 + \left(\frac{v}{w+u}\right)^2} \frac{1}{(w+u)} \qquad \Rightarrow \quad \frac{\partial F}{\partial v}(1,0,1) = -\frac{1}{2}$$

$$\frac{\partial F}{\partial w}(u,v,w) = -1 - \frac{1}{1 + \left(\frac{v}{w+u}\right)^2} \frac{-v}{(w+u)^2} \qquad \Rightarrow \quad \frac{\partial F}{\partial w}(1,0,1) = -1$$

Pro derivace implicitně zadané funkce g tedy

$$\frac{\partial g}{\partial v}(0,1) = -\frac{\dfrac{\partial F}{\partial v}(1,0,1)}{\dfrac{\partial F}{\partial u}(1,0,1)} = -\frac{-\frac{1}{2}}{1} = \frac{1}{2}$$

$$\frac{\partial g}{\partial w}(0,1) = -\frac{\dfrac{\partial F}{\partial w}(1,0,1)}{\dfrac{\partial F}{\partial u}(1,0,1)} = -\frac{-1}{1} = 1,$$

odkud

$$\mathrm{d}g(0,1,\mathrm{d}v,\mathrm{d}w) = \frac{1}{2}\mathrm{d}v + \mathrm{d}w$$

a

$$g(0.01, 0.98) \doteq g(0,1) + \mathrm{d}g(0,1; 0.01, -0.02) = 1 + \frac{0.01}{2} - 0.02 = 1 + 0.005 - 0.02 = 0.985.$$

Výše spočtené derivace lze spočíst samozřejmě i parciálním derivováním definující rovnice

$$g(v,w) = w + \operatorname{arctg}\frac{v}{w+g(v,w)}$$

$$\frac{\partial g}{\partial v}(v,w) = \frac{1}{1+\left(\frac{v}{w+g(v,w)}\right)^2}\cdot\frac{w+g(v,w)-v\frac{\partial g}{\partial v}(v,w)}{(w+g(v,w))^2}$$

$$\Rightarrow \quad \frac{\partial g}{\partial v}(0,1) = 1\cdot\frac{2-0}{2^2} = \frac{1}{2}$$

$$\frac{\partial g}{\partial w}(v,w) = 1 + \frac{1}{1+\left(\frac{v}{w+g(v,w)}\right)^2}\cdot\frac{-v\left(1+\frac{\partial g}{\partial w}(v,w)\right)}{(w+g(v,w))^2} \Rightarrow \quad \frac{\partial g}{\partial v}(0,1) = 1$$

4. příklad

$$\int \cos x \mathrm{d}x = \sin x, \qquad \int \cos^2 x \mathrm{d}x = \int \frac{1+\cos(2x)}{2}\mathrm{d}x = \frac{x}{2} + \frac{\sin(2x)}{4}$$

$$\int \cos^3 x \mathrm{d}x = \int \cos^2 x \cdot \cos x \mathrm{d}x = \int (1-\sin^2 x)\cdot\cos x \mathrm{d}x = \int (1-t^2)\mathrm{d}t = t - \frac{t^3}{3} = \sin x - \frac{\sin^3 x}{3}$$

$$\int \cos x + \cos^2 x + \cos^3 x \mathrm{d}x = \frac{x}{2} + 2\sin x + \frac{\sin(2x)}{4} - \frac{\sin^3 x}{3}$$

$$\int_0^{\frac{\pi}{2}} \cos x + \cos^2 x + \cos^3 x \mathrm{d}x = \frac{\pi}{4} + 2 - \frac{1}{3} = \frac{\pi}{4} + \frac{5}{3}$$